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Brief paper

A dwell time approach for the stabilization of mixed Continuous/Discrete switched systems ☆

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ABSTRACT

In this paper, we study the stabilization properties of Continuous/Discrete switched systems using the dwell time approaches. The considered system switches between a continuous-time subsystem evolving on time intervals with variable lengths and a discrete-time subsystem with variable discrete step sizes. Since the time domain is non-uniform, and the stability of the discrete-time subsystem with variable discrete step sizes does not depend only on the dynamic of the system, but also on the length of the discrete steps, the usual conditions of dwell time approaches may not be applicable to stabilize this special class of switched systems. Motivated by that, stabilizing dwell time and average dwell time conditions are derived by introducing the time scales theory and numerical examples are proposed to illustrate the effectiveness of the proposed methods.

1. Introduction

A wide range of physical and engineering systems involve coupling between continuous dynamics and discrete events. Systems in which these two kinds of dynamics interact are called switched systems. Switched systems consist of a finite number of different modes subject to a discrete rule that orchestrates the switching law between them. The stability issue is the main concern in the field of switched systems, which have been widely studied in the literature and have attracted much attention (Decarlo et al., 2000; Liberzon, 2003; Lin & Antsaklis, 2009). It is known that even if the switched system is fully composed of stable modes, it is still possible to have a divergent trajectories caused by the failure to absorb the energy increase caused by the switching, except for some special cases, under some algebraic conditions (Narendra & Balakrishnana, 1994; Zhai et al., 2006). On the other hand, in the presence of unstable modes, if one either stays too long at or switches too frequently to the unstable subsystem, stability may be lost. In these cases, the switched system can be stabilized under an appropriate switching law. The appropriate switching signals in time domain is determined by the dwell-time or average dwell-time switching. It is shown that if one switches less frequently, one may trade off the energy increase caused by switching (or unstable modes), and maintain the stability of the system, which means that the dwell time (or in average) between any consecutive switching have to be no smaller than a constant $\tau_a > 0$. This concept has been studied for continuous-time switched systems, extended to the case where both stable and unstable subsystems coexist (Hespanha & Morse, 1999; Liberzon, 2003; Xiang & Xiao, 2014; Zhaoa et al., 2012). The dwell time results were extended to the discrete-time switched systems in Geromel and Colaneri (2006), Zhai et al. (2002), Zhang et al. (2014), Ren et al. (2017). Most of the existing results are concerned with switched systems operating on the continuous or discrete uniform time domains separately. However, in several areas of engineering applications, there are many situations when the switched system is composed of continuous-time and discrete-time subsystems, such as communication failures in networked control systems, where the information is exchanged over some disconnected time intervals due to unreliable communication channels. A cascaded system composed of a continuous-time plant, a set of discrete-time controllers and switching between them is also an example. In this situation the time domain is neither *continuous* nor *uniformly discrete*. To extend the existing results for systems evolving on a non-uniform time domain, time scales theory was introduced. This theory unifies between continuous-time and discrete-time analysis that allows to study the stability and control of dynamical systems on an arbitrary time domain (discrete with variable steps or a combination between discrete and continuous times) (Bohner

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& Peterson, 2001). The stability of dynamic equations on time scales has been investigated in Potzsche et al. (2003), Peterson and Raffoul (2005), Du and Tien (2007). Switched systems on an arbitrary time scales have been studied and generalized in Davis et al. (2010), Eisenbarth et al. (2014). In this work, we are interested in a special class of switched systems, where the system switches between continuous-time subsystems evolving on variable length intervals and discrete-time subsystems with variable discrete-step sizes. Such time domain is non-uniform, therefore, it is necessary to derive new conditions to establish stability for this class of switched systems by introducing the time scale $\mathbb{T}$, formed by a union of disjoint closed intervals with variable lengths and gaps. This special class of switched systems was considered in Taousser et al. (2014), Taousser et al. (2015b), Taousser et al. (2015a), where conditions have been derived to guarantee the exponential stability. This class of switched systems has been introduced in the study of the problem of multi-agent systems with intermittent information transmissions in Taousser et al. (2016), and in the modeling of intermittent hormone therapy for prostate cancer in Higgins et al. (2020). However, in all these works, the time scale $\mathbb{T}$ is supposed to be known and given in advance. Motivated by that, we are interested in deriving a stabilizing switching rule of this class of switched systems via dwell time conditions. Notice that, when the discrete step size is variable in time, the stability of the discrete-time subsystem depends strongly on the size of the discrete steps, which should be confined by a pair of upper and lower bounds to guarantee its stability. In this case, the existing results of dwell time approaches cannot be applied. Therefore, it is necessary to derive new dwell time conditions to establish stability for this class of switched systems by introducing the time scales.

In this paper, new stabilizing switching law are derived using dwell time approaches in time scales theory. Numerical examples show the effectiveness of the proposed methods, and an application to a consensus problem for multi-agent systems with intermittent information transmission is provided.

2. Preliminaries and problem statement

2.1. Preliminaries on time scale theory

In this subsection, we recall some basics on time scale theory (see Bohner and Peterson (2001)) and derive a proposition to characterize the time scale exponential function.

A *time scale* $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$. We define the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$, and the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ by $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$. The mapping $\mu : \mathbb{T} \to \mathbb{R}^+$, called the *graininess function*, is defined by $\mu(t) = \sigma(t) - t$, which measure the distance between two consecutive times. A point $t \in \mathbb{T}$ is called *right-scattered* if $\sigma(t) > t$, *right-dense* if $\sigma(t) = t$, *left-scattered* if $\rho(t) < t$ and *left-dense* if $\rho(t) = t$. The set $\mathbb{T}^\kappa$ is defined as follows: if $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^\kappa = \mathbb{T} - \{m\}$; otherwise $\mathbb{T}^\kappa = \mathbb{T}$. Let $f : \mathbb{T} \to \mathbb{R}$, the Δ-derivative of f at $t \in \mathbb{T}^\kappa$ is defined as

$$f^\Delta(t) = \lim_{s \to t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s}.$$

One can notice, if $\mathbb{T} = \mathbb{R}$, $\sigma(t) = t$ and $f^\Delta(t) = \dot{f}(t)$, which is the euclidian derivative of f; and if $\mathbb{T} = h\mathbb{Z}$, $\sigma(t) = t + h$, then $f^\Delta(t) = \frac{f(t+h)-f(t)}{h}$. So using time scale theory, the theory of differential and difference equations is unified. A function $f : \mathbb{T} \to \mathbb{R}$ is said to be *rd-continuous*, if it is continuous at right-dense points in $\mathbb{T}$ and its left limit exists at left-dense points in $\mathbb{T}$. A function $p : \mathbb{T} \to \mathbb{R}$ is *regressive* if $1 + \mu(t)p(t) \neq 0, \forall t \in \mathbb{T}^\kappa$. We denote the set of regressive and rd-continuous functions by $\mathcal{R}$ and by $\mathcal{R}^+$, if they satisfy $1 + \mu(t)p(t) > 0, \forall t \in \mathbb{T}^\kappa$ (i.e., *positively regressive functions*). Similarly, a function matrix $A : \mathbb{T} \to \mathbb{R}^n$ is called *regressive*, if and only if $(I + \mu(t)A)$ is invertible $\forall t \in \mathbb{T}$, or equivalently A is regressive if and only if all its eigenvalues are regressive. The set $\mathcal{R}$ together with the *circle addition* $\oplus$ defined by $(p \oplus q)(t) = p(t) + q(t) + \mu(t)p(t)q(t),\ p, q \in \mathcal{R},\ t \in \mathbb{T}$ is an Abelian group. The inverse element is $\ominus p(t) = \frac{-p(t)}{1+\mu(t)p(t)}$ and the *circle subtraction* is defined by $(p \ominus q)(t) = p(t) \oplus (\ominus q(t)) = \dfrac{p(t) - q(t)}{1 + \mu(t)q(t)}$. Note that if $p, q \in \mathcal{R}$, then $\ominus p$, $p \oplus q$, $p \ominus q$, $q \ominus p \in \mathcal{R}$.

For $h > 0$ let the $\mathbb{C}_h := \{z \in \mathbb{C} :\ z \neq \frac{-1}{h}\}$, and for $h = 0$, $\mathbb{C}_0 := \mathbb{C}$. Define, for $z \in \mathbb{C}_h$, the function $\xi_h(z) := \frac{1}{h}\log(1 + hz)$, and for $h = 0$, $\xi_0(z) := z$. The *generalized exponential function* of $p \in \mathcal{R}$ on the time scale $\mathbb{T}$ is expressed by

$$e_p(t, s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right) \quad \text{for} \quad s, t \in \mathbb{T},$$

and the Δ-integral is used (see Bohner and Peterson (2001)). In particular, for $\mathbb{T} = \mathbb{R}$, $e_p(t, t_0) = e^{\int_s^t p(\tau)\Delta\tau}$ and for $\mathbb{T} = h\mathbb{Z}$, $e_p(t, s) = \prod_{\tau=s}^{t-h}(1 + hp(\tau))$. The exponential function has the properties, for $p, q \in \mathcal{R}$, $t, s \in \mathbb{T}$:

$$e_p(t, s) = \frac{1}{e_p(s, t)} = e_{\ominus p}(s, t), \quad e_p(t, s)e_q(t, s) = e_{p\oplus q}(t, s).$$

Let $A \in \mathbb{R}^{n\times n}$ be a regressive matrix, the generalized exponential function $e_A(t, t_0)x_0$ is the unique solution of

$$x^\Delta(t) = Ax(t), \quad x(t_0) = x_0, \quad t_0 \in \mathbb{T}. \tag{1}$$

System (1) is exponentially stable on $\mathbb{T}$, if there exists a constant $\beta \geq 1$ and $0 > \lambda \in \mathcal{R}^+$, such that the corresponding solution satisfies

$$\|x(t)\| \leq \beta\|x_0\|e_\lambda(t, t_0), \quad \forall t \in \mathbb{T}.$$

This characterization is a generalization of the definition of exponential stability on $\mathbb{R}$ or $h\mathbb{Z}$. More specifically, the condition $0 > \lambda \in \mathcal{R}^+$ is reduced to $\lambda < 0$ for $\mathbb{T} = \mathbb{R}$, and to $0 < 1 + \mu(t)\lambda < 1$, $\forall t \in \mathbb{T}$ for any discrete time scale $\mathbb{T}$ with graininess function $\mu(t)$. Since $e_\lambda(t, t_0)$ can be negative, the positive regressivity of λ is needed (see Bohner and Peterson (2001)). To study the stability of linear dynamical systems on time scale $\mathbb{T}$, a particular open set of the complex plane called the *Hilger circle* is defined for all $t \in \mathbb{T}$ as

$$\mathcal{H}_{\mu(t)} := \left\{z \in \mathbb{C} :\ |1 + z\mu(t)| < 1,\ z \neq -\frac{1}{\mu(t)}\right\}.$$

When $\mu(t) = 0$, we define $\mathcal{H}_0 = \{z \in \mathbb{C} : \Re(z) < 0\} = \mathbb{C}^-$, the open left-half complex plane. The smallest Hilger circle (denoted $\mathcal{H}_{\min}$) is the Hilger circle associated with $\mu(t) = \mu_{\max} = \sup_{t\in\mathbb{T}} \mu(t)$. A regressive constant matrix A is called Hilger stable if $\operatorname{spec}(A) \subset \mathcal{H}_{\min}$ (i.e., all eigenvalues of A are in $\mathcal{H}_{\min}$) (Gard & Hoffacker, 2003).

Theorem 1 (*Potzsche et al., 2003*).

Let a regressive constant matrix $A \in \mathbb{R}^{n\times n}$. There exists an invertible matrix $Q \in \mathbb{C}^{n\times n}$ such that, the generalized exponential function of A is given by

$$e_A(t, s) = Q\begin{pmatrix} e_{J_1}(t, s) & & \\ & \ddots & \\ & & e_{J_l}(t, s)\end{pmatrix}Q^{-1}, t, s \in \mathbb{T}^\kappa,$$

such that, for $k = 1, 2, \ldots, l \leq n$,

$$e_{J_k}(t, s) = e_{\lambda_k}(t, s)\begin{pmatrix} 1 & m^1_{\lambda_k}(t, s) & \ldots & m^{n_k-1}_{\lambda_k}(t, s) \\ & 1 & \ldots & m^{n_k-2}_{\lambda_k}(t, s) \\ & & \ddots & \vdots \\ & & & 1\end{pmatrix},$$

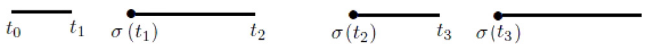

Fig. 1. $\mathbb{T} = \mathbb{P}_{\{\sigma(t_k),t_{k+1}\}} = \cup_{k=0}^{\infty}[\sigma(t_k), t_{k+1}]$.

where $n_1 + \cdots + n_l = n$. The mappings $m^n_{\lambda_k} : \mathbb{T} \times \mathbb{T}^\kappa \to \mathbb{C}$, called monomials of degree n, are recursively defined by

$$m^0_{\lambda_k}(t,s) = 1 \;;\; m^{n+1}_{\lambda_k}(t,s) = \int_s^t \frac{m^n_{\lambda_k}(\tau,s)}{1+\mu(\tau)\lambda_k(\tau)}\Delta\tau.$$

For $\mathbb{T} = \mathbb{R}$, one gets $m^n_{\lambda_k}(t,s) = \frac{(t-s)^n}{n!}$, $t, s \in \mathbb{R}$.

For $\mathbb{T} = h\mathbb{Z}$, we have $m^n_{\lambda_k}(t,s) = \frac{(t-s)^n}{n!(1+h\lambda_k)^n}$, $t, s \in \mathbb{T}$. If λ is uniformly regressive, (i.e. there exists a $\gamma > 0$ such that $\gamma^{-1} \le |1+\mu(t)\lambda|$, $\forall t \in \mathbb{T}^\kappa$), then the estimate

$|m^n_\lambda(t,s)| \le \gamma^n(t-s)^n$ holds for $t \ge s$, $n \in \mathbb{N}$ (see Potzsche et al. (2003) for more details).

2.2. Problem statement

This paper is devoted to the study of switched system $x^\Delta(t) = A_i x(t)$, where $A_i \in \mathbb{R}^{n\times n}$ and $x(t) \in \mathbb{R}^n$ is the state vector. The switching instants are expressed by the sequence $\{t_0, t_1, \sigma(t_1), t_2, \sigma(t_2), t_3, \ldots, \sigma(t_k), t_k, \ldots\}$, without finite accumulation points, where $\sigma(.)$ is the forward jump operator, with $\sigma(t_0) = t_0$, $\sigma(t_k) > t_k$, $\forall k \in \mathbb{N}^*$ (see Fig. 1). The considered system switches between two modes such that, $i \in \{c, d\}$ ={continuous, discrete}, where A_c is activated at instants $\sigma(t_k)$ and A_d is activated at t_k. Time scales theory is introduced to study the stability of this special class of switched systems on $\mathbb{T} = \mathbb{P}_{\{\sigma(t_k),t_{k+1}\}} = \cup_{k=0}^{\infty}[\sigma(t_k), t_{k+1}]$, such that

$$x^\Delta(t) = \begin{cases} A_c x(t), & \text{for} \quad t \in \cup_{k=0}^{\infty}[\sigma(t_k), t_{k+1}[\\ A_d x(t), & \text{for} \quad t \in \cup_{k=0}^{\infty}\{t_{k+1}\} \end{cases} \tag{2}$$

Note that, the second equation is the discrete-time linear dynamic which corresponds to state jumps for instance, during a variable period of time $\mu(t_k) = \sigma(t_k) - t_k$, $k \in \mathbb{N}^*$. The objective is to design $\mathbb{T} = \cup_{k=0}^{\infty}[\sigma(t_k), t_{k+1}]$ as a control parameter stabilizing the switched system (2), in the presence of unstable modes, according to conditions that will be derived in the next Section.

The eigenvalues of A_c (resp. A_d) are denoted by λ^j_c (resp. λ^j_d). It is known that, if all eigenvalues of A_c have a negative real part, so the continuous-time subsystem is stable. However, the stability of the discrete-time subsystem with variable discrete step sizes, depends on the eigenvalues of A_d and also on $\mu(t)$ at each instant $t \in \cup_{k=0}^{\infty}\{t_{k+1}\}$. Note that A_d is Hilger stable, if all eigenvalues λ^j_d of A_d lie strictly within the Hilger circle $\mathcal{H}_{\min}$. This means that

$$|1+\mu(t_k)\lambda^j_d| < 1, \;\forall 1 \le j \le n, \;\forall k \in \mathbb{N}^*. \tag{3}$$

Condition (3) implies that the values $\mu(t_k)$ have to satisfy $0 < \mu(t_k) < \gamma_d$, $\forall k \in \mathbb{N}^*$, where $\gamma_d = \min_{1\le j\le n}\left\{\frac{-2\Re(\lambda^j_d)}{|\lambda^j_d|^2}\right\}$, with $\Re(.)$ denote the real part and $|.|$ is the modulus.

On the other hand, A_d is unstable if it has at least one eigenvalue λ^j_d such that $\Re(\lambda^j_d) > 0$ or if all the eigenvalues of A_d have a negative real part, but there exists at least one eigenvalue λ^j_d such that $|1+\mu_{\min}\lambda^j_d| > 1$.

3. Main results

We shall study the stabilization problem of switched system (2), via dwell time switching. The cases where unstable modes exist will be considered. First, we need the following proposition:

Proposition 1. *Let $\mathbb{T}$ be an arbitrary time scale with graininess function $\mu(.)$, and let a regressive matrix $A \in \mathbb{R}^{n\times n}$ with eigenvalues λ_k, for $k = 1, \ldots, l \le n$ be given. Let λ be an eigenvalue of A such that $|e_\lambda(t,s)| = \max_{1\le k\le l}|e_{\lambda_k}(t,s)|$, $\forall t, s \in \mathbb{T}^\kappa$. For every $\alpha \in \mathcal{R}^+$, which satisfies $e_\alpha(t,s) \ge |e_\lambda(t,s)|$, $\forall t, s \in \mathbb{T}^\kappa$, $t \ge s$, there exists $\beta \ge 1$, such that*

$$\|e_A(t,s)\| \le \beta\; e_\alpha(t,s), \quad \forall t \ge s.$$

Proof. From Theorem 1, the upper bound of the generalized exponential matrix $e_A(t,s)$ is given by

$$\|e_A(t,s)\| \le \|Q\|\|Q^{-1}\|\;|e_\lambda(t,s)| \times \left(1+\max_{1\le k\le l}\left(\max_{1\le n\le n_k-1}|m^n_{\lambda_k}(t,s)|\right)\right),$$

for $t, s \in \mathbb{T}^\kappa$, $t \ge s$. For a small $\varepsilon \ge 0$, with $\varepsilon \oplus \lambda \ne 0$, let us define the constant α, satisfying $e_\alpha(t,s) = |e_{\lambda\oplus\varepsilon}(t,s)|$, such that

$$\|Q\|\|Q^{-1}\|\;|e_\lambda(t,s)|(1+\max_{1\le k\le l}(\max_{1\le n\le n_k-1}|m^n_{\lambda_k}(t,s)|))$$

$$\le \beta\;|e_{\lambda\oplus\varepsilon}(t,s)| = \beta\; e_\alpha(t,s),$$

for some positive constant β. Hence

$$\beta \ge \|Q\|\|Q^{-1}\|(1+\max_{1\le k\le l}(\max_{1\le n\le n_k-1}|m^n_{\lambda_k}(t,s)|))$$

$$\times\;|e_{\lambda\ominus(\lambda\oplus\varepsilon)}(t,s)|$$

$$= \|Q\|\|Q^{-1}\|\;(1+\max_{1\le k\le l}(\max_{1\le n\le n_k-1}|m^n_{\lambda_k}(t,s)|))$$

$$\times\; e_{\ominus\varepsilon}(t,s). \tag{4}$$

Note that $e_{\ominus\varepsilon}(t,s) = e^{-\int_s^t \frac{\log(1+\mu(\tau)\varepsilon)}{\mu(\tau)}\Delta\tau}$ is always decreasing in time, since $\varepsilon > 0$, and the above term is always bounded. Since (4) holds, $\forall t \ge s$, $\exists\, \beta \ge 1$ with

$$\beta = \max_t(\|Q\|.\|Q^{-1}\| \times(1+\max_{1\le k\le l}(\max_{1\le n\le n_k-1}|m^n_{\lambda_k}(t,s)|))\;e_{\ominus\varepsilon}(t,s)),$$

such that $\|e_A(t,s)\| \le \beta\; e_\alpha(t,s)$. ■

Remark 1. If A is diagonalizable, then $\|e_A(t,s)\| \le \beta\; e_\alpha(t,s)$ with $e_\alpha(t,s) = |e_\lambda(t,s)|$ and $\beta = \|Q\|\|Q^{-1}\|$.

3.1. Stabilization via dwell time

Consider the switched system (2) evolving on $\mathbb{T} = \cup_{k=0}^{\infty}[\sigma(t_k), t_{k+1}]$, such that there always exist the constants α_c, $\alpha_d \in \mathcal{R}^+$, $\beta_c \ge 1$ and $\beta_d \ge 1$, satisfying, from Proposition 1, the following inequalities:

- For all $t, s \in \cup_{k=0}^{\infty}[\sigma(t_k), t_{k+1}[$, $s \le t$,

$$\|e_{A_c}(t,s)\| = \|e^{A_c(t-s)}\| \le \beta_c\; e_{\alpha_c}(t,s) = \beta_c e^{\alpha_c(t-s)} \tag{5}$$

- For all $k \in \mathbb{N}^*$,

$$\begin{aligned}\|e_{A_d}(\sigma(t_k),t_k)\| = \|I+\mu(t_k)A_d\| &\le \beta_d\; e_{\alpha_d}(\sigma(t_k),t_k) \\ &= \beta_d(1+\mu(t_k)\alpha_d).\end{aligned} \tag{6}$$

Remark 2. If A_c and A_d are diagonalizable, so $\alpha_c = \max_{1\le j\le n}\Re(\lambda^j_c)$ and α_d can be computed such that, $\max_{1\le j\le n}|1+\mu(t)\lambda^j_d| = (1+\mu(t)\alpha_d)$, $\forall t \in \cup_{k=0}^{\infty}\{t_{k+1}\}$. The constants $\beta_c = \|Q_c\|.\|Q_c^{-1}\|$ and $\beta_d = \|Q_d\|.\|Q_d^{-1}\|$. If A_c and A_d are not diagonalizable, so $\alpha_c = \max_{1\le j\le n}(\Re(\lambda^j_c)) + \varepsilon$ and α_d can be computed as $\max_{1\le j\le n}|1+\mu(t)(\lambda^j_d \oplus \varepsilon)| = (1+\mu(t)\alpha_d)$, $\forall t \in \cup_{k=0}^{\infty}\{t_{k+1}\}$, for some small $\varepsilon > 0$. From Proposition 1, and definition of monomials of the Jordan matrix, β_c and β_d can be computed as

$$\beta_c = \max_t(\|Q_c\|.\|Q_c^{-1}\|(1+t^{(n-1)})\, e^{-\varepsilon(t-s)}),$$

$$\beta_d = \max_t(\|Q_d\|.\|Q_d^{-1}\| \left(1+\left|\frac{\mu(t)}{1+\mu(t)\lambda_k}\right|^{(m-1)}\right) \frac{1}{(1+\mu(t)\varepsilon)}),$$

where n and m are the highest geometric multiplicity corresponding to the eigenvalue λ_k of A_c and A_d respectively. Note that, if $\alpha_c < 0$, then A_c is Hilger stable and if $0 > \alpha_d \in \mathcal{R}^+$ (i.e.; $0 < 1+\mu(t_k)\alpha_d < 1,\ \forall k \in \mathbb{N}^*$), then A_d is Hilger stable.

To determine the dwell time conditions for stability of switched system (2), we compute its general solution.

For $\sigma(t_k) \le t < t_{k+1}$ and $\sigma(t_0) = t_0 = 0$, the solution of system (2) is given by (see Taousser et al. (2014)),

$$\begin{aligned} x(t) &= e_{A_c}(t,\sigma(t_k))\, e_{A_d}(\sigma(t_k),t_k)\, e_{A_c}(t_k,\sigma(t_{k-1})) \\ &\quad \times \cdots e_{A_d}(\sigma(t_1),t_1)\, e_{A_c}(t_1,t_0)\, x_0. \\ &= e^{A_c(t-\sigma(t_k))}\,(I+\mu(t_k)A_d)\, e^{A_c(t_k-\sigma(t_{k-1}))} \\ &\quad \times \cdots (I+\mu(t_1)A_d)\, e^{A_c t_1}\, x_0. \end{aligned} \tag{7}$$

According to (5), (6), an upper bound of (7) is given by

$$\begin{aligned} \|x(t)\| &\le \|e^{A_c(t-\sigma(t_k))}\|\,\|(I+\mu(t_k)A_d)\|\,\|e^{A_c(t_k-\sigma(t_{k-1}))}\| \\ &\quad \times \cdots \|(I+\mu(t_1)A_d)\|\,\|e^{A_c t_1}\|\ \|x_0\| \\ &\le \beta_c e^{\alpha_c(t-\sigma(t_k))}\ \beta_d(1+\mu(t_k)\alpha_d)\ \beta_c e^{\alpha_c(t_k-\sigma(t_{k-1}))} \\ &\quad \times \cdots \beta_d(1+\mu(t_1)\alpha_d)\ \beta_c e^{\alpha_c t_1}\ \|x_0\| \\ &\le e^{\sum_{i=0}^{k}[\log(\beta_c)+\alpha_c(t_{i+1}-\sigma(t_i))+\log(\beta_d(1+\mu(t_i)\alpha_d))]} \\ &\quad \times \|x_0\|. \end{aligned} \tag{8}$$

It is assumed throughout the paper that $\mathbb{T}$ is unbounded above and the graininess function is bounded (i.e.; $\mu_{\min} \le \mu(t_k) \le \mu_{\max}, \forall k \in \mathbb{N}^*$). Denote by $\tau_k := t_{k+1} - \sigma(t_k),\ \forall k \in \mathbb{N}$, the duration of each continuous-time subsystem which is assumed to be bounded (i.e.; $\tau_{\min} \le \tau_k \le \tau_{\max},\ \forall k \in \mathbb{N}$). The aim of the paper is to design a time scale $\mathbb{T} = \cup_{k=0}^{\infty}[\sigma(t_k), t_{k+1}]$ which stabilizes the switched system (2). Note that, in order for the solution of the switched system to be well defined, we have to choose a time scale $\mathbb{T}$ which ensures the regressivity of the matrices A_c and A_d. This means that we have to design $\mathbb{T}$ with $\mu(t_k)$ satisfying $|1+\mu(t_k)\lambda_d^j| \neq 0,\ \forall k \in \mathbb{N}^*,\ \forall 1 \le j \le n$. Note that A_c is always regressive on the intervales $[\sigma(t_k), t_{k+1}]$. In what follows, A_d is Hilger stable means that $\mu(t_k),\ \forall k \in \mathbb{N}^*$ of the time scale $\mathbb{T}$ will be designed to satisfy condition (3). Note that, even if the two subsystems are Hilger stable the switched system (2) may be unstable. For that, we will derive dwell time conditions which stabilize the switched system (2) in the following Theorems.

Theorem 2. *Let α_c, α_d, β_c and β_d be defined as in (5), (6). If one of the following conditions is satisfied:*

(i) A_c and A_d are Hilger stable and $\mathbb{T}$ is designed such that,

$$\tau_k > \frac{-\log(\beta_c)}{\alpha_c}, \quad \frac{1-\beta_d}{\alpha_d\ \beta_d} < \mu(t_k) < \frac{-1}{\alpha_d}, \quad \forall k \in \mathbb{N}. \tag{9}$$

or

$$\tau_k > \frac{-\log(\beta_c\beta_d)}{\alpha_c}, \quad 0 < \mu(t_k) < \frac{-1}{\alpha_d},\ \forall k \in \mathbb{N}, \tag{10}$$

with $\alpha_c < 0$ and $0 > \alpha_d \in \mathcal{R}^+$.

(ii) A_c is Hilger stable, A_d is unstable, and $\mathbb{T}$ is designed such that,

$$\tau_k > \frac{-\log(\beta_d\beta_c)}{\alpha_c}, \quad 0 < \mu(t_k) < \frac{e^{-\alpha_c\tau_{\min}}}{\alpha_d\beta_d\beta_c} - \frac{1}{\alpha_d},\ \forall k \in \mathbb{N}, \tag{11}$$

with $\alpha_c < 0$ and $\alpha_d > 0$.

(iii) A_c is unstable, A_d is Hilger stable, and $\mathbb{T}$ is designed such that,

$$\frac{1-\beta_c\beta_d}{\alpha_d\beta_c\beta_d} < \mu(t_k) < \frac{-1}{\alpha_d},\ \forall k \in \mathbb{N}^*. \tag{12}$$

$$0 < \tau_k < \frac{-\log(\beta_c\beta_d(1+\mu_{\min}\alpha_d))}{\alpha_c}, \quad \forall k \in \mathbb{N}, \tag{13}$$

with $\alpha_c > 0$ and $0 > \alpha_d \in \mathcal{R}^+$.

Then the switched system (2) is exponentially stable.

Proof. Let $\mu(t_k),\ \forall k \in \mathbb{N}^*$ satisfy (3) and the regressivity of A_d. The upper bound of the solution of (2) is given by

$$\|x(t)\| \le e^{\sum_{i=0}^{k}[\log(\beta_c)+\alpha_c\tau_i+\log(\beta_d(1+\mu(t_i)\alpha_d))]}\|x_0\|. \tag{14}$$

(i) Since A_c is stable, so $\alpha_c < 0$, and from (9), we get

$$\sum_{i=0}^{k}[\log(\beta_c)+\alpha_c\tau_i] \le k[\log(\beta_c)+\alpha_c\tau_{\min}] < 0.$$

Also, A_d is Hilger stable, so $0 > \alpha_d \in \mathcal{R}^+$, which means that $0 < (1+\mu(t_k)\alpha_d) < 1,\ \forall k \in \mathbb{N}^*$, and from (9), we get

$$\sum_{i=i}^{k}\log[\beta_d(1+\mu(t_i)\alpha_d)] \le k\log[\beta_d(1+\mu_{\min}\alpha_d)] < 0.$$

From (14), and the above inequalities, an upper bound for $x(t)$ is given by

$$\|x(t)\| \le e^{k[(\log(\beta_c)+\alpha_c\tau_{\min})+\log(\beta_d(1+\mu_{\min}\alpha_d))]}\ \|x_0\|. \tag{15}$$

The terms at the exponential function are negative, which implies that the solution converges exponentially to zero when $k \to \infty\ \ (t \to \infty)$.

We can derive another dwell time conditions which are more restrictive for the continuous-time subsystem and less restrictive for the discrete-time subsystem. From (14), the upper bound of the solution can be written as

$$\|x(t)\| \le e^{\sum_{i=0}^{k}[(\log(\beta_c\beta_d)+\alpha_c\tau_i)+\log(1+\mu(t_i)\alpha_d)]}\|x_0\|. \tag{16}$$

From (10), all the terms of the exponential in (16) are negative, and the solution is exponentially stable.

(ii) Suppose that A_c is stable and A_d is unstable (i.e.; $\mu(t_k),\ \forall k \in \mathbb{N}^*$ does not satisfy condition (3)). From (16), we have

$$\|x(t)\| \le e^{\sum_{i=0}^{k}[\alpha_c\tau_i+\log(\beta_c\beta_d(1+\mu(t_i)\alpha_d))]}\|x_0\|,$$

such that, $\alpha_c < 0$ and $\alpha_d > 0$. So we get

$$\|x(t)\| \le e^{k[(\alpha_c\tau_{\min})+\log(\beta_c\beta_d(1+\mu_{\max}\alpha_d))]}\|x_0\|. \tag{17}$$

From condition (11), we get

$$1 < (1+\mu(t_i)\alpha_d) < \frac{e^{-\alpha_c\tau_{\min}}}{\beta_c\beta_d},\ \forall 1 \le i \le k.$$

which implies that all the terms at the exponential in (17) are negative and the solution converges exponentially to zero. Note that, in order for the above inequality to be satisfied, $\tau_{\min}$ must satisfy the condition $\tau_{\min} > \frac{-\log(\beta_d\beta_c)}{\alpha_c}$, which conclude the proof.

(iii) From (12) we get, $0 < \beta_c\beta_d(1+\mu(t_k)\alpha_d) < 1$, and from (13), we have $\alpha_c\tau_k + \log(\beta_c\beta_d(1+\mu_{\min}\alpha_d)] < 0,\ \forall k \in \mathbb{N}$, such that, $\alpha_c > 0$. From (16), the upper bound of $x(t)$ is given by

$$\begin{aligned} \|x(t)\| &\le e^{\sum_{i=0}^{k}[\alpha_c\tau_i+\log(\beta_c\beta_d(1+\mu(t_i)\alpha_d))]}\ \|x_0\| \\ &= e^{k[\alpha_c\tau_{\max}+\log(\beta_c\beta_d(1+\mu_{\min}\alpha_d))]}\ \|x_0\|. \end{aligned}$$

The terms of the exponential are negative, which implies that the solution of the switched system (2) converges exponentially to zero when $k \to \infty\ \ (t \to \infty)$. ∎

Remark 3. Note that, conditions (11) and (12) in assumptions (ii) and (iii) can be relaxed as follows:

(ii) For $\alpha_c < 0$ and $\alpha_d > 0$, and for each $\tau_k > \frac{-\log(\beta_d\beta_c)}{\alpha_c}$, we have $0 < \mu(t_k) < \frac{e^{-\alpha_c\tau_k}}{\alpha_d\beta_d\beta_c} - \frac{1}{\alpha_d}, \; \forall k \in \mathbb{N}$,

(iii) For $\alpha_c > 0$ and $\alpha_d < 0$, and for each $\mu(t)$ satisfying $\frac{1-\beta_c\beta_d}{\alpha_d\beta_c\beta_d} < \mu(t_k) < \frac{-1}{\alpha_d}, \; \forall k \in \mathbb{N}^*$, we have $0 < \tau_k < \frac{-\log(\beta_c\beta_d(1+\mu_k\alpha_d))}{\alpha_c}, \quad \forall k \in \mathbb{N}^*$.

This conditions are less restrictive but require more computations.

3.2. Stabilization via average dwell time

In this section we will show that there may exist some consecutive switching subsystems in (2) which not satisfy the dwell time conditions determined in Section 3.1, but the switched system (2) can be stabilized if certain dwell time conditions are satisfied in average for each consecutive switched subsystem. Let us define the stabilizing average dwell time conditions for continuous and discrete switched systems respectively.

Definition 1 (*Hespanha & Morse, 1999*; *Zhai et al., 2002*)**.** We say that τ_a is an average dwell time for the continuous switched system $\dot{x}(t) = A_i x(t)$ (resp. the discrete switched system $x(k+1) = A_i x(k)$), if for some positive number N_0 (called the chatter bound) and for all $t \geq \tau \geq 0$, the number of switching signals over the time interval $[\tau, t)$ (resp. on the interval $[k', k)$), denoted by $N_\sigma(\tau, t)$ (resp. $N_\sigma(k', k)$), satisfies

$$N_\sigma(\tau, t) \leq N_0 + \frac{t-\tau}{\tau_a}.$$

(resp. $N_\sigma(k', k) \leq N_0 + \frac{k-k'}{\tau_a}, \quad k' \in \mathbb{N}, \; k' \leq k$).

Which means that in average, the dwell time between any two consecutive switching is no smaller than τ_a.

Note that, the above average dwell time conditions cannot be used for stability of the class of switched systems (2). Next, we will derive a new average dwell time conditions for the switched systems (2), in the presence of unstable modes. In the following, we consider $\alpha_c, \alpha_d, \beta_c$ and β_d defined as in (5) and (6), and let $\beta = \max\{\beta_c, \beta_d\} \geq 1$. Let $N(0, t)$ be the number of switching over the interval $[0, t], \;\; \forall\, t \in \cup_{k=0}^{\infty}[\sigma(t_k), t_{k+1}]$. Denote the total duration time of the continuous-time subsystem (resp. the discrete-time subsystem) from $t_0 = 0$ to $t, \;\; \forall t \in \cup_{k=0}^{\infty}[\sigma(t_k), t_{k+1}]$, by $T_c(0, t)$ (resp. $T_d(0, t)$).

Theorem 3. *If the following assumptions are fulfilled:*

(i) A_c and A_d are Hilger stable such that

$$0 < \mu(t_k) < \frac{-1}{\alpha_d} \;\; \text{with} \;\; \alpha_d < 0, \quad \forall k \in \mathbb{N}^*. \tag{18}$$

(ii) For a given $\lambda > 0$, an arbitrary $N_0 > 0$ and $\forall t \in \cup_{k=0}^{\infty}[\sigma(t_k), t_{k+1}]$,

$$N(0, t) \leq N_0 + \frac{t}{\tau_a^*}, \quad \text{with} \quad \tau_a^* = \frac{\log(\beta)}{\lambda}. \tag{19}$$

Then, the switched system (2) is exponentially stable if the average dwell time between any two consecutive switching is greater than τ_a^*, such that (18) is satisfied.

Proof. Let $\mu(t_k), \;\; \forall k \in \mathbb{N}^*$ satisfy (3) and the regressivity of A_d. From the upper bound of the solution of system (2), for $\sigma(t_k) \leq t \leq t_{k+1}, \;\; \forall k \in \mathbb{N}$, expressed in (8), we get

$$\begin{aligned} \|x(t)\| &\leq \beta^{2k+1} e^{\alpha_c(t-\sum_{i=0}^{k}\mu(t_i))} \prod_{i=1}^{k}(1+\mu(t_i)\alpha_d)\, \|x_0\| \\ &\leq \beta^{2k+1} e^{\alpha_c(t-\sum_{i=0}^{k}\mu(t_i))} (1+\mu_{\min}\alpha_d)^k \, \|x_0\| \\ &= \beta\, e^{N(0,t)\log(\beta)+\alpha_c(t-\sum_{i=0}^{k}\mu(t_i))+k\log(1+\mu_{\min}\alpha_d)} \\ &\quad \times \|x_0\|. \end{aligned} \tag{20}$$

We have $k \geq \dfrac{\sum_{i=0}^{k}\mu(t_i)}{\mu_{\max}}$. Hence, one gets

$$\begin{aligned} &\|x(t)\| \leq \beta\, \|x_0\| \\ &\quad \times e^{N(0,t)\log(\beta)+\alpha_c(t-\sum_{i=0}^{k}\mu(t_i))+\sum_{i=0}^{k}\mu(t_i)\left(\frac{\log(1+\mu_{\min}\alpha_d)}{\mu_{\max}}\right)} \end{aligned} \tag{21}$$

Fixing $\mu_{\min}$ and $\mu_{\max}$ satisfying (18). Let $\lambda' = \max\{\alpha_c, \frac{\log(1+\mu_{\min}\alpha_d)}{\mu_{\max}}\} < 0$, so we get

$$\|x(t)\| \leq \beta\, e^{N(0,t)\log(\beta)+\lambda' t}\, \|x_0\|. \tag{22}$$

System (2) is exponentially stable with a desired rate of decrease λ^*, such that $\lambda' < \lambda^* < 0$, if $e^{N(0,t)\log(\beta)+\lambda' t} \leq e^{\gamma+\lambda^* t}$, for an arbitrary constant $\gamma > 0$, which implies that $N(0, t)\log(\beta) \leq \gamma + (-\lambda' + \lambda^*)t$. So we get, $N(0, t) \leq N_0(t, 0) + \dfrac{t}{\tau_a^*}$, with $N_0(t, 0) = [\dfrac{\gamma}{\log(\beta)}] > 0$ and $\tau_a^* = \dfrac{\log(\beta)}{-\lambda' + \lambda^*}$, where $[.]$ denotes the integer part. Which concludes the proof for $\lambda = (-\lambda' + \lambda^*)$. ■

Remark 4. If $\beta = 1$, the term in the exponential in (22) is always negative, and system (2) is exponentially stable under an arbitrary switching such that (18) is satisfied.

Theorem 4. *If the following assumptions are fulfilled:*

(i) A_c is stable and A_d is unstable.

(ii) Let the switching law stability

$$\frac{T_c(0, t)}{T_d(0, t)} \geq \frac{-\frac{\log(1+\mu_{\max}\alpha_d)}{\mu_{\min}} + \lambda^*}{\alpha_c - \lambda^*}, \tag{23}$$

for given constants λ, λ^* such that $\alpha_c < \lambda^* < \lambda < 0$.

(iii) For an arbitrarly $N_0 > 0$ and $\forall t \in \cup_{k=0}^{\infty}[\sigma(t_k), t_{k+1}]$,

$$N(0, t) \leq N_0 + \frac{t}{\tau_a^*}, \;\; \text{with} \;\; \tau_a^* = \frac{\log(\beta)}{\lambda - \lambda^*}, \tag{24}$$

Then, the switched system (2) is exponentially stable with rate of decrease λ for any average dwell time, between two consecutive switching greater than τ_a^*.

Proof. Since A_d is supposed to be unstable, so $\alpha_d > 0$. From (20), and according to (i), we have for a fixed $\mu_{\max}$ and $\mu_{\min}$, the following upper bound for the solution

$$\begin{aligned} \|x(t)\| &\leq \beta e^{N(t,0)\log(\beta)+\alpha_c(t-\sum_{i=0}^{k}\mu(t_i))+k\log(1+\mu_{\max}\alpha_d)} \\ &\quad \times \|x_0\|, \end{aligned} \tag{25}$$

with $\alpha_c < 0$. We have $k \leq \frac{\sum_{i=1}^{k}\mu(t_i)}{\mu_{\min}}$, so

$$\begin{aligned} &\|x(t)\| \leq \beta\, \|x_0\| \\ &\quad \times e^{N(t,0)\log(\beta)+\alpha_c(t-\sum_{i=0}^{k}\mu(t_i))+\sum_{i=0}^{k}\mu(t_i)\frac{\log(1+\mu_{\max}\alpha_d)}{\mu_{\min}}} \\ &\quad = \beta\, e^{N(t,0)\log(\beta)+\alpha_c T_c(0,t)+T_d(0,t)\frac{\log(1+\mu_{\max}\alpha_d)}{\mu_{\min}}} \;\; \|x_0\|. \end{aligned}$$

Let λ^* with $\alpha_c < \lambda^* < 0$, such that condition (23) is satisfied, which is equivalent to

$$\alpha_c T_c(0, t) + T_d(0, t)\frac{\log(1+\mu_{\max}\alpha_d)}{\mu_{\min}} < \lambda^* t.$$

So an upper bound of $x(t)$ is given by

$$\|x(t)\| \leq \beta\ e^{N(0,t)\log(\beta)+\lambda^* t}\ \|x_0\| \tag{26}$$

Let λ be the desired rate of decay of the switched system (2) such that $\lambda^* \leq \lambda < 0$, so $e^{N(0,t)\log(\beta)+\lambda^* t} \leq e^{\gamma+\lambda t}$, for an arbitrary constant $\gamma > 0$, which is equivalent to $N(0,t) \leq N_0 + \frac{t}{\tau_a^*}$, with $N_0 = [\frac{\gamma}{\log(\beta)}]$ and $\tau_a^* = \frac{\log(\beta)}{\lambda-\lambda^*}$. ■

Remark 5. If $\beta = 1$ and condition (23) is satisfied, so the switched system (2) is exponentially stable.

Theorem 5. *If the following assumptions are fulfilled:*

(i) A_c is unstable and A_d is Hilger stable, such that

$$0 < \mu(t_k) < \frac{-1}{\alpha_d}, \text{ with } \alpha_d < 0, \quad \forall k \in \mathbb{N}^*, \ \alpha_c > 0. \tag{27}$$

(ii) Suppose that for a fixed $\mu_{\min}$ and $\mu_{\max}$ satisfying (27), and for a given negative constants λ, λ^*, such that

$$\frac{\log(1+\mu_{\min}\alpha_d)}{\mu_{\max}} < \lambda^* < \lambda < 0, \tag{28}$$

the switching law is determined as

$$\frac{T_d(0,t)}{T_c(0,t)} \geq \frac{\alpha_c - \lambda^*}{-\frac{\log(1+\mu_{\min}\alpha_d)}{\mu_{\max}} + \lambda^*}, \tag{29}$$

(iii) For an arbitrary $N_0 > 0$, and $\forall t \in \cup_{k=0}^{\infty}[\sigma(t_k), t_{k+1}[$,

$$N(0,t) \leq N_0 + \frac{t}{\tau_a^*}, \quad \text{with} \quad \tau_a^* = \frac{\log(\beta)}{\lambda - \lambda^*}, \tag{30}$$

Then the switched system (2) is exponentially stable with rate of decrease λ and for an average dwell time between any two consecutive switching greater than τ_a^*.

Proof. Similarly to the above analysis, and according to (i) we have, for $\sigma(t_k) \leq t \leq t_{k+1}$, $k \in \mathbb{N}$

$$\|x(t)\| \leq \beta\ e^{N(0,t)\log(\beta)+\alpha_c T_c(0,t)+T_d(0,t)\frac{\log(1+\mu_{\min}\alpha_d)}{\mu_{\max}}}\|x_0\|, \tag{31}$$

for a fixed $\mu_{\min}$ and $\mu_{\max}$ satisfying (27) and $\alpha_c > 0$. Let $\frac{\log(1+\mu_{\min}\alpha_d)}{\mu_{\max}} < \lambda^* < 0$. Condition (29) is equivalent to

$$\alpha_c T_c(0,t) + T_d(0,t)\frac{\log(1+\mu_{\min}\alpha_d)}{\mu_{\max}} < \lambda^* t, \text{ so}$$

$$\|x(t)\| \leq \beta\ e^{N(0,t)\log(\beta)+\lambda^* t}\ \|x_0\|$$

Let λ be the rate of decrease of the trajectories of system (2), such that, $\lambda^* \leq \lambda < 0$, so $N(0,t)\log(\beta) + \lambda^* t \leq \gamma + \lambda t$, for an arbitrary constant $\gamma > 0$. Which implies that $N(0,t) \leq N_0 + \frac{t}{\tau_a^*}$, for $N_0 = [\frac{\gamma}{\log(\beta)}]$ and $\tau_a^* = \frac{\log(\beta)}{\lambda-\lambda^*}$, and conclude the proof. ■

Remark 6. If $\beta = 1$ and conditions (27), (29) are satisfied, so the switched system (2) is exponentially stable.

4. Numerical results

Consider the switched system (2) with $A_c = \begin{pmatrix} 2 & 1 \\ -1 & -0.5 \end{pmatrix}$ which is unstable, such that $\lambda_c^1 = 0$, $\lambda_c^2 = 1.5 = \alpha_c$, with $\beta_c = 3$. Let $A_d = \begin{pmatrix} \frac{-13}{4} & \frac{-1}{2} \\ \frac{1}{2} & \frac{-1}{2} \end{pmatrix}$, such that $\lambda_d^1 = -3.1559 = \alpha_d$, $\lambda_d^2 = -0.5941$ and $\beta_d = 1.4639$. The matrix A_d is regressive and stable if $0 < \mu(t) < 0.6337$ and $\mu \neq 0.3169$. From assumption (iii) of Theorem 2, the dwell time of the discrete-time subsystem is $0.2447 < \mu < 0.3169$. Take for example, $\mu(t) = 0.3$, the switched system is stable if the dwell time of each continuous-time subsystem $\tau_k < 0.9689, \forall k \in \mathbb{N}$. For $\tau_k = 0.9$, the switched system is stable as shown in Fig. 2. For $\tau_k = 2$, the continuous-time condition of stability is not satisfied and the switched system diverges Fig. 3. For $\mu(t) = 0.1$, $\mu(t) = 0.5$ and $\tau_k = 0.9$, A_d is still stable, but the corresponding dwell time condition of stability is not satisfied and the switched system diverges Figs. 4, 5. For the average dwell time, let $\mu_{\min} = 0.2$, $\mu_{\max} = 0.5$, $\lambda^* = -0.9$ and $\lambda = -0.3046$, so $\tau_a^* = 1.8452$ and $\frac{T_c}{T_d} \geq 2.1920$. If we activate the switched system on $\mathbb{T} = \cup_{k=0}^{\infty}[0.9k + \frac{5k}{10k+15}, 0.9(k+1)]$, the average dwell time conditions are satisfied and the system is stable Fig. 6.

5. Application to consensus problem under intermittent information transmission

To illustrate the viability of the proposed scheme, we investigate the consensus problem for linear multi-agent systems (MASs) with intermittent information transmissions. We will show that this problem can be converted to a switched system between a continuous-time subsystem (when the communication occur) and a discrete-time subsystem (when the communication fails). Consider MAS consisting of N agents whose model is described by the following linear dynamics,

$$\dot{x}_i = Ax_i + Bu_i, \quad \dot{x}_0 = Ax_0 \qquad i \in \{1, \ldots, N\} \tag{32}$$

where $x_0 \in \mathbb{R}^n$ is the state of the leader, $x_i \in \mathbb{R}^n$ is the state of agent i and $u_i \in \mathbb{R}^m$ is the control input of agent i. $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are constant real matrices. The communication network among the N agents is described by a graph $\mathcal{G}$ which consists of a node set $\mathcal{V} = \{1, 2, \ldots, N\}$ and an edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. Each edge $(i,j) \in \mathcal{E}$ in the directed graph (Ren & Beard, 2005), corresponds to the information link between agent i and agent j. The graph $\mathcal{G}$ is represented by the adjacency matrix $\mathcal{G} = (a_{ij}) \in \mathbb{R}^{N\times N}$ defined by $a_{ij} = 1$ if $(j,i) \in \mathcal{E}$ and $a_{ij} = 0$, otherwise. The Laplacian matrix of $\mathcal{G}$ is defined as $H = (m_{ij}) \in \mathbb{R}^{N\times N}$ with $m_{ii} = \sum_{j=1}^{N} a_{ij}$ and $m_{ij} = -a_{ij}$ for $i \neq j$. Hereafter, suppose that each communication failure has a bounded duration (denoted by $\mu_{\max} \in \mathbb{R}^+$), the pair (A, B) is stabilizable, and the graph $\mathcal{G}$ is fixed and directed. Let z_i be the local information available for agent i, such that

$$z_i = \sum_{j\in\mathcal{N}_i}(x_j - x_i) + d_i(x_0 - x_i), \tag{33}$$

where $\mathcal{N}_i$ is the set of neighbors of agent i such that $a_{ij} = 1$ with $d_i = 1$, if the leader state is available to follower i and with $d_i = 0$ otherwise.

It is assumed that local information is exchanged between neighboring agents through a communication channel over some disconnected time intervals because of possible sensor failures or communication obstacles, such that the agents can communicate with their neighbors over the time intervals $\cup_{k=0}^{\infty}[\sigma(t_k), t_{k+1}[$. At t_{k+1}, the communication fails during a period $\mu(t_{k+1}) = \sigma(t_{k+1}) - t_{k+1}$. Based on the available local information, the following distributed intermittent controller is proposed, $\forall i \in \{1, \ldots, N\}$,

$$u_i(t) = \begin{cases} Kz_i(t), & \text{if } t \in \cup_{k=0}^{\infty}[\sigma(t_k), t_{k+1}[\\ Kz_i(t_{k+1}), & \text{if } t \in \cup_{k=0}^{\infty}[t_{k+1}, \sigma(t_{k+1})[, \end{cases} \tag{34}$$

such that, over the time intervals $[t_{k+1}, \sigma(t_{k+1})[$ the feedback control does not evolve due to the absence of local information. The state error between the leader and the agent i is determined

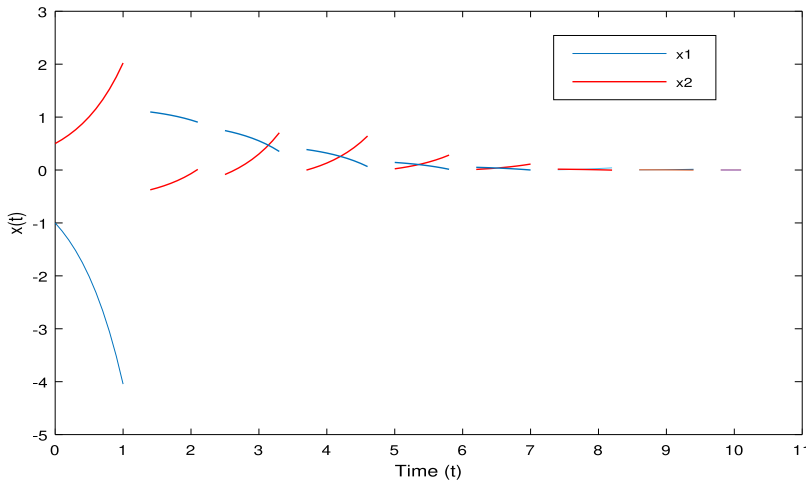

Fig. 2. Stable trajectories for $\mu = 0.3$, $\tau_k = 0.9$.

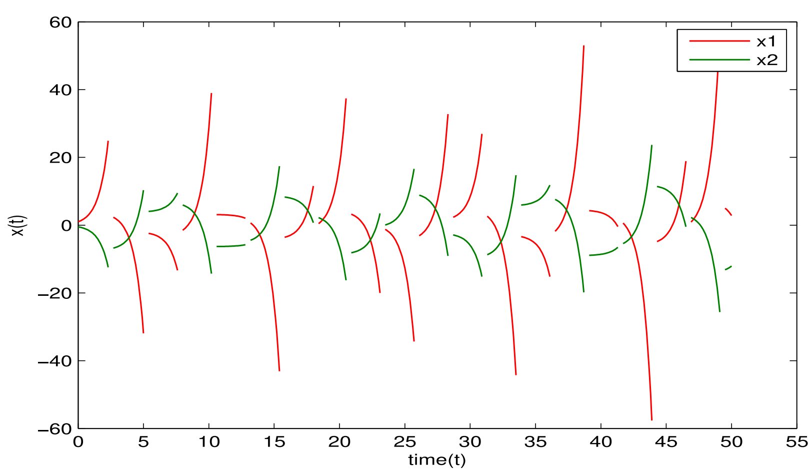

Fig. 3. Unstable trajectories for $\mu = 0.3$, $\tau_k = 2$.

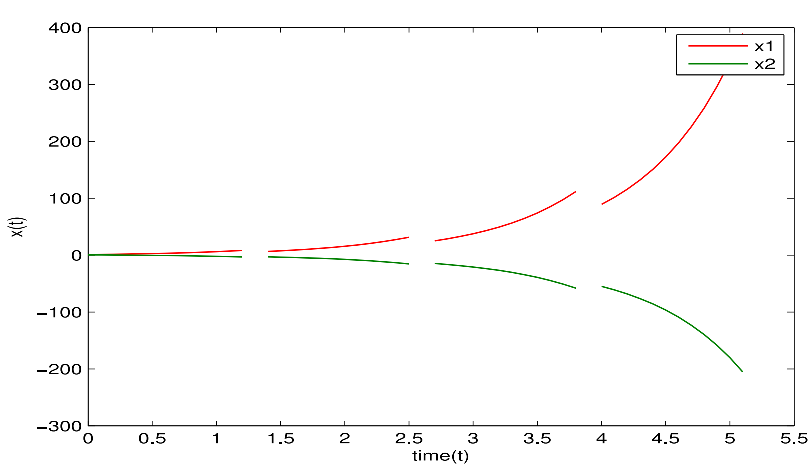

Fig. 4. Unstable trajectories for $\mu = 0.1$, $\tau_c = 0.9$.

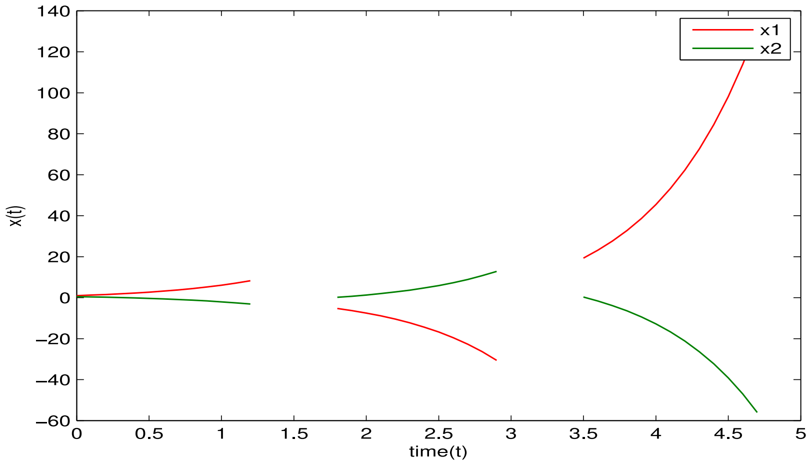

Fig. 5. Unstable trajectories for $\mu = 0.5$, $\tau_c = 0.9$.

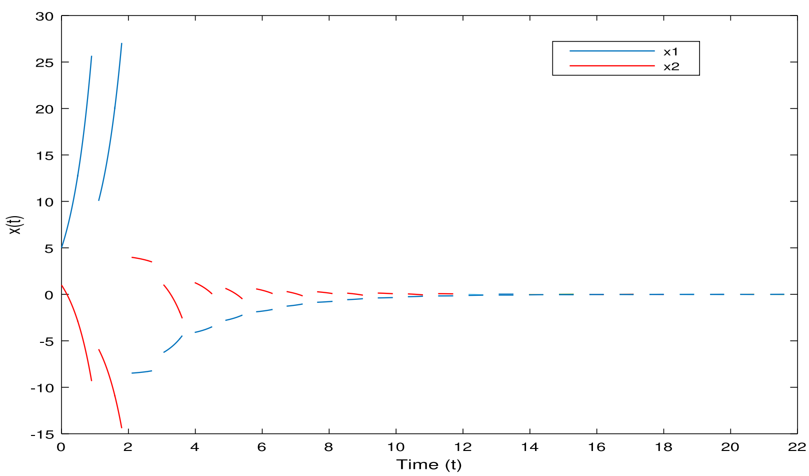

Fig. 6. Unstable trajectories for $\mu = 0.5$, $\tau_c = 0.9$.

by $e_i = x_i - x_0$. Let $u = (u_1^T, \ldots, u_N^T)^T$. The dynamic of the tracking error $e = (e_1^T, \ldots, e_N^T)^T$ can be written in the compact form as

$$\dot{e}(t) = (I_N \otimes A)e(t) + (I_N \otimes B)u(t),$$
$$u(t) = \begin{cases} -(H \otimes K)e(t), & \text{if } t \in \cup_{k=0}^{\infty}[\sigma(t_k), t_{k+1}[\\ -(H \otimes K)e(t_{k+1}), & \text{if } t \in \cup_{k=0}^{\infty}[t_{k+1}, \sigma(t_{k+1})[\end{cases} \tag{35}$$

The closed-loop system (35) becomes

$$\dot{e} = \begin{cases} [(I_N \otimes A) - (H \otimes BK)]e(t), \\ \qquad t \in \cup_{k=0}^{\infty}[\sigma(t_k), t_{k+1}[, \\ (I_N \otimes A)e(t) - (H \otimes BK)e(t_{k+1}), \\ \qquad t \in \cup_{k=0}^{\infty}[t_{k+1}, \sigma(t_{k+1})[. \end{cases} \tag{36}$$

Using the definition of the Δ-derivative and considering the specific time scale $\mathbb{T} = \cup_{k=0}^{\infty}[\sigma(t_k), t_{k+1}]$, the closed-loop system (36) can be converted to the switched system, as in (2) (see Taousser et al. (2015b, 2016) for more details):

$$e^{\Delta}(t) = \begin{cases} [(I_N \otimes A) - (H \otimes BK)]e(t), \\ \qquad t \in \cup_{k=0}^{\infty}[\sigma(t_k), t_{k+1}[\\ \left(\dfrac{e^{(I_N \otimes A)\mu(t)} - I}{\mu(t)}\right)[I_{nN} - (H \otimes A^{-1}BK)]e(t), \\ \qquad t \in \cup_{k=0}^{\infty}\{t_{k+1}\} \end{cases} \tag{37}$$

The objective is to design the time scale $\mathbb{T}$, such that the state error e_i between the leader and agent i satisfy $\lim_{t\to\infty} \|e_i(t)\| = 0$, $\forall i \in \{1, \ldots, N\}$ (i.e., the system (32) still stable, even when we lose the communication between agents for some period of time). That the leader–follower consensus problem is equivalent to the stabilization problem of the switched system (37) by designing the time scale $\mathbb{T} = \cup_{k=0}^{\infty}[\sigma(t_k), t_{k+1}]$, according to the dwell time conditions derived in Sections 3.1 and 3.2. This, will enable us to achieve exponential consensus under intermittent information transmissions while avoiding the derivation of complex Lyapunov–Krasovskii and Razumikhin functions.

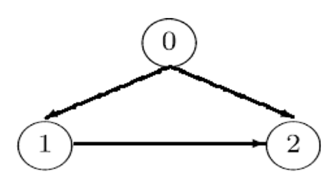

Fig. 7. Communication topology.

Remark 7. Notice that, if A is not invertible, we can always determine the discrete matrix via the convergence power series $\mathcal{E}(A\mu(t)) = \sum_{n=1}^{\infty} \frac{(A\mu(t))^{n-1}}{n!}$, and $A_d = \left(\dfrac{e^{(I_N \otimes A)\mu(t)} - I}{\mu(t)}\right)[\mathcal{E}(A\mu(t))(H \otimes BK)]$.

To illustrate the procedure, let us consider the MAS which consists of one leader and 2 followers satisfying the communication topology shown in Fig. 7. Let $A = \begin{pmatrix} 0 & 1 \\ 0.1 & 0.05 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

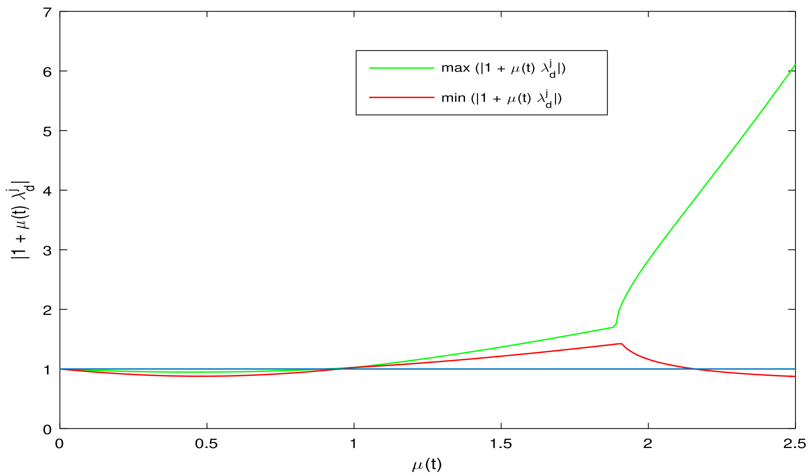

Fig. 8. Values of $\mu(t)$ for stability of A_d.

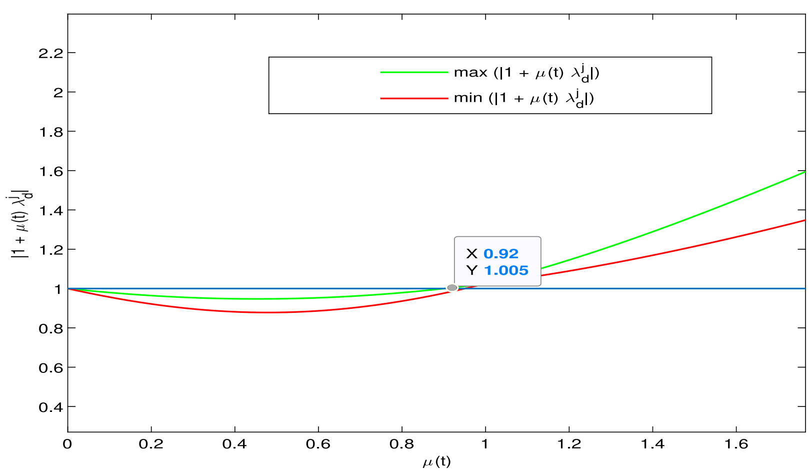

Fig. 9. Values of μ for stability of A_d.

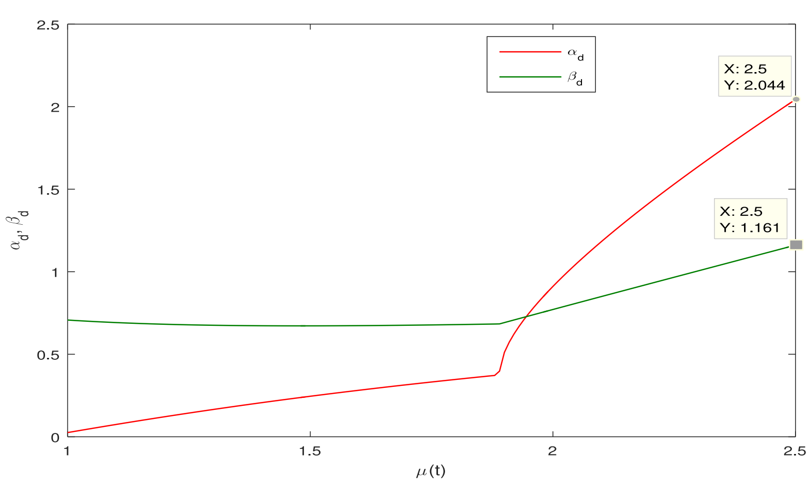

Fig. 10. α_d and β_d for $1 \leq \mu \leq 2.5$.

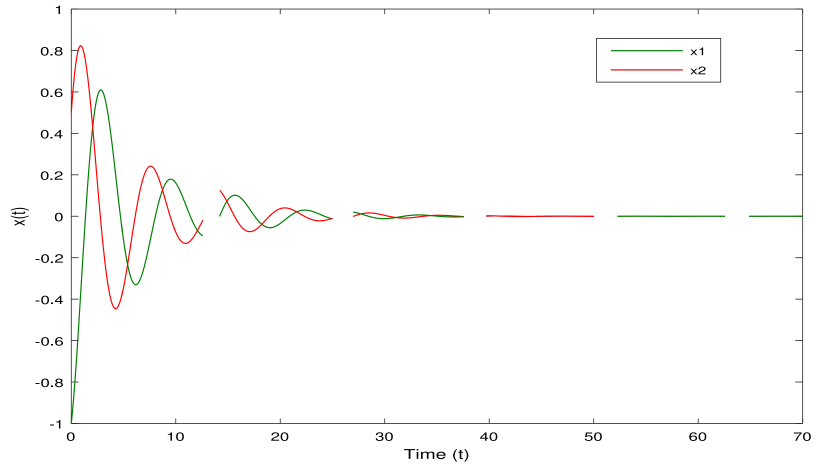

Fig. 11. Stable trajectories, with $x_0 = [-1,\ 0.5,\ -1,\ 1]^T$.

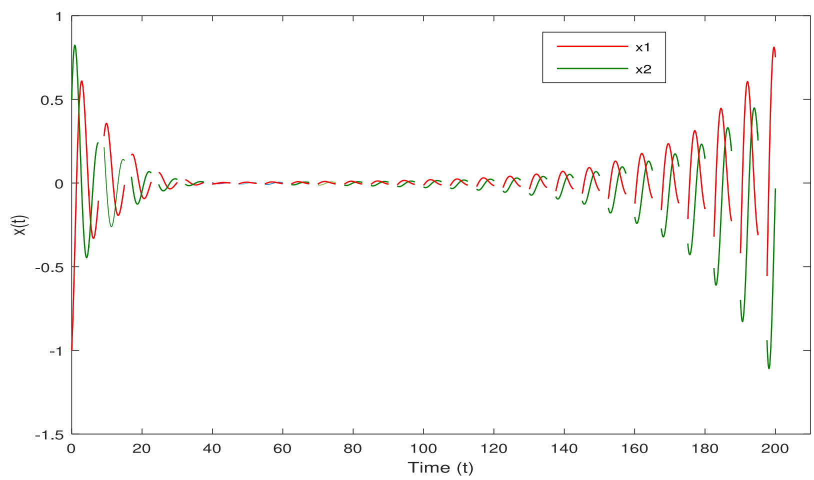

Fig. 12. Unstable trajectories, with $x_0 = [-1,\ 0.5,\ -1,\ 1]^T$.

and the weighted matrix $H = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}$. The control gain is set as $K = \begin{pmatrix} 1 & 0.5 \end{pmatrix}$, with $A_c = [(I_2 \otimes A) - (H \otimes BK)]$, which is stable, such that $\lambda_c^{1,2} = -0.475 \pm j1.294$, $\lambda_c^{3,4} = -0.225 \pm j0.9216$ (we have $\alpha_c = -0.225$, $\beta_c = 3.4397$), and $A_d = \left(\frac{e^{(I_2 \otimes A)\mu(t)} - I}{\mu(t)}\right) [I_4 - (H \otimes A^{-1}BK)]$. The matrix A_d is Hilger stable, if $|1 + \mu(t)\lambda_d^j| < 1, \forall 1 \leq j \leq 4$. As shown in Fig. 8 and its zoom Fig. 9, the discrete-time subsystem is stable if $0 < \mu(t) < 0.92$. Let us choose $1 \leq \mu(t) \leq 2.5$, which leads to the instability of A_d. As in Proposition 1, α_d can be chosen such that, $(1 + \mu(t)\alpha_d) \geq \max_j |1 + \mu(t)\lambda_d^j|$, which is plotted in Fig. 10. Let $\alpha_d = 2.044$, so β_d can be chosen, such that $\beta_d \geq \frac{\|e_{A_d}(\sigma(t_k), t_k)\|}{e_{\alpha_d}(\sigma(t_k), t_k)} = \frac{\|I_4 - \mu(t)A_d\|}{(1+\mu(t)\alpha_d)}$. From Fig. 10, let $\beta_d = 1.161$. The dwell time conditions for stability are

$$\tau_k > \frac{-\log(\beta_c\beta_d)}{\alpha_c} = 6.1541, \quad 1 \leq \mu(t) \leq \frac{e^{-\alpha_c \tau_{\min}}}{\alpha_d\beta_d\beta_c} - \frac{1}{\alpha_d} = 2.5.$$

Consider the time scale $\mathbb{T} = \bigcup_{k=0}^{\infty}[12.64 + \frac{5k}{2k+1.33}, 12.64(k+1)]$, which satisfies the stability conditions, as shown in Fig. 11, the switched system is stable. If we change the time scale $\mathbb{T} = \bigcup_{k=0}^{\infty}[5.5 + \frac{10k}{2k+1.33}, 5.5(k+1)]$ by decreasing τ_k, such that the condition of dwell time is not satisfied, the system becomes unstable (see Fig. 12).

6. Conclusion

A special class of switched systems between a continuous-time and a discrete-time subsystems with variable discrete steps has been considered. By introducing time scales theory, dwell time conditions are derived to stabilize the switched system in the presence of unstable modes. The conditions provide a new method to exploit the stabilization of this particular class of switched systems. The traditional dwell time conditions for stability of switched systems do not apply for the considered class, so that, time scales theory provides to be the right mathematical tool to solve such problem.

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