A Study on Energy Preservability of Runge-Kutta Methods in Power System Simulation

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***Abstract*—Runge-Kutta methods have been widely used in power system time-domain simulations. However, conventional Runge-Kutta methods can not preserve the total energy of the simulated system because they are not symplectic integrators. For the explicit Euler method and fourth-order Runge-Kutta method, this paper first finds explicit formulae on how the total energy of the simulated system trajectory can change with the integration time step by using the Hamiltonian system formulation of a single-machine-infinite-bus system. The formulae discover the existence of a critical time step for energy-preserving simulation. Then, the formulae are used to evaluate the error in observed damping of the system as well as the correction if the simulation is conducted for an extended period with a time step different from the critical time step. Finally, the formulae are applied to Kundur's two-area four-generator system regarding its dominant mode.**

***Keywords*—Time-domain simulation, symplectic integrator, energy preservability, Hamiltonian system.**

I. Introduction

Time-domain power system simulation is critical in assessing dynamic behaviors and transient stability of a power system subject to a disturbance [1]-[4]. Explicit Runge-Kutta (R-K) methods are adopted in most simulation tools for solving an initial value problem of the ordinary differential equations (ODEs) of a power system model for a given contingency. In general, a small integration time step is needed to avoid numerical instability and limit errors in simulation results over a desired simulation period. However, when simulation needs to be conducted for an extended period, e.g. for a power grid emulator or testbed system that needs to conduct numerical simulation for minutes to hours [5],[6], simulation errors can accumulate and their influences on the accuracy of simulation can become more significant. One example on a result of such errors is the incredible damping ratio estimated directly from the numerical simulation result since the errors can introduce negative or positive damping; another example is that thegrowth of the system total energy may lead to the numerical instability of the simulation. A fundamental cause of the error accumulation is that R-K methods are not structure-preserving numerical solvers or in other words, they do not fall into the type of symplectic integrators which can preserve the structural information, e.g. the total energy, of the simulated system [3]. Although symplectic variants of R-K methods have been proposed in literature for more accurate results of extended-term simulations, these variants are often implicit and are slower than conventional explicit R-K methods. There have been very few papers concerning the error analysis of conventional R-K methods in power system simulation if they fail to preserve the total system energy.

This paper studies whether a power system can preserve its total energy when simulated by two most widely used explicit R-K methods, the Euler method and the fourth-order R-K (R-K 4) method. It is discovered that the total energy cannot be preserved for all cases with the Euler method and most cases of the R-K 4 method unless a critical time step is adopted. By using a Hamiltonian system formulation of the single-machine-infinite-bus (SMIB) system, the paper finds explicit formulae on how the total system energy can change with the integration time step. The concept of "numerical damping" is adopted to define the fake damping, positive or negative, added to the simulation result if the simulation fails to preserve the energy [8]. Sometimes numerical damping can eliminate high frequency modes to stabilize the system in structural dynamics problems [9]. However, in power system simulations this may introduce fake damping for modal analysis which should be avoided. Using the formulae, errors in simulations are analyzed to predict the numerical instability. Also, a correction that cancels numerical damping can be achieved towards finding the true damping of the system from non-energy-preserving simulation results. Also, appropriate integration steps are suggested for extended-term simulations [10],[11] in which energy preservation is desired. Finally, because the SMIB system can be considered as the system equivalent regarding inter-area oscillations of a two-area power system. The formulae and conclusions on the SMIB system are extended and tested on Kundur's two-area four-generator system.

The rest of this paper is organized as follows. Section II derives and studies the formulae on the change of the total energy for an undamped SMIB system simulated by the Euler method and R-K 4 method, proposes numerical damping, and discusses its correction for the SMIB system. Section III presents case studies on both the SMIB system and the two-area system. Conclusions are summarized in section IV.

II. Energy Preservation with R-K Methods on an SMIB System

This section will adopt a Hamiltonian system formulation of the SMIB system to find formulae on how its total energy changes with the integration step. Both linearized and nonlinear models are studied on the SMIB system subject to small and large disturbances. Then, the numerical damping is proposed and corrected toward an estimate of the true damping of the system simulated by the R-K 4 method.

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A. Considering a small disturbance

Subject to a small disturbance, the SMIB system without damping can be approximated by its linearized model, i.e. a second-order harmonic oscillator system described by:

$$M\ddot{x} + Kx = 0 \tag{1}$$

where x is the state variable on its rotor angle deviation, M is the inertia and K is the positive recovering force constant. Rewrite (1) into the form (2) of a Hamiltonian system about the position q and momentum p:

$$\begin{cases} \dot{p} = -Kq \\ \dot{q} = \dfrac{p}{M}, \end{cases} \tag{2}$$

where

$$\begin{cases} p = M\dot{x} \\ q = x \end{cases}. \tag{3}$$

Without damping, the total energy of the system should be unchanged and equal to the sum of kinetic and potential energies:

$$E = \frac{p^2}{2M} + \frac{Kq^2}{2}. \tag{4}$$

To solve (2), first consider the Euler method with integration step h. The total energy E can be calculated by (5) as well as its change after each iteration from N to N+1 by (6):

$$\begin{cases} p_{N+1} = p_N - hKq_N \\ q_{N+1} = q_N - h\dfrac{p_N}{M} \end{cases} \tag{5}$$

$$\begin{aligned} \Delta E_{N+1} &= E_{N+1} - E_N \\ &= \frac{p_{N+1}^2}{2M} + \frac{Kq_{N+1}^2}{2} - \frac{p_N^2}{2M} - \frac{Kq_N^2}{2} \\ &= \frac{-hKq_N(2p_N - hKq_N)}{2M} + \frac{hKp_N(2q_N + \frac{hp_N}{M})}{2M} \\ &= h^2\frac{K}{M}(\frac{p_N^2}{2M} + \frac{Kq_N^2}{2}) \\ &= h^2\frac{K}{M}E_N. \end{aligned} \tag{6}$$

From (6), the total energy at time t is

$$E_t = (1 + h^2\frac{K}{M})^{t/h}E_0, \tag{7}$$

where

$$t = t_0 + N \times h = N \times h. \tag{8}$$

Obviously, a physical system, e.g. the linearized SMIB system, has both K and M be positive. This means that the total energy manifested from the simulation result will increase rapidly with the Euler method. Thus, the simulation result presents negative damping even if the true system has no damping.

Next, if the R-K 4 method is adopted, the following steps in (9) can be taken to obtain formulae on the change of energy at each iteration in (10) and the energy as a function of time in (11).

$$\begin{aligned} k_1 &= h \times f(p_N, q_N) \\ k_2 &= h \times f((p_N, q_N) + 0.5k_1) \\ k_3 &= h \times f((p_N, q_N) + 0.5k_2) \\ k_4 &= h \times f((p_N, q_N) + k_3) \\ (p_{N+1}, q_{N+1}) &= (p_N, q_N) + \frac{k_1 + 2k_2 + 2k_3 + k_4}{6} \end{aligned} \tag{9}$$

$$\begin{aligned} \Delta E_{N+1} &= E_{N+1} - E_N \\ &= \frac{p_{N+1}^2}{2M} + \frac{Kq_{N+1}^2}{2} - \frac{p_N^2}{2M} - \frac{Kq_N^2}{2} \\ &= -h^6K^3(\frac{-Kh^2 + 8M}{576M^4})(\frac{p_N^2}{2M} + \frac{Kq_N^2}{2}) \\ &= -h^6K^3(\frac{-Kh^2 + 8M}{576M^4})E_N \end{aligned} \tag{10}$$

$$E_t = (1 - h^6K^3(\frac{-Kh^2 + 8M}{576M^4}))^{t/h}E_0. \tag{11}$$

Unlike the Euler method, the R-K 4 method for $K > 0$ and $M > 0$ has a critical time step h_c by which simulation can preserve the total energy:

$$h_c = \sqrt{\frac{8M}{K}}, \tag{12}$$

where

$$\Delta E_{N+1}(h_c) = E_{N+1} - E_N = 0. \tag{13}$$

If $h < h_c$, the total energy will decrease at every iteration, which exhibits positive damping even if the system itself has no damping; if $h > h_c$, the total energy will increase rapidly to present negative damping and finally cause a numerically unstable simulation.

The largest one-step change of energy can be found as the stationary point with respect to h. Namely,

$$\frac{d\Delta E_{N+1}}{dh} = \frac{-K^3h^5E_N}{72M^4} \times (Kh^2 - 6M) = 0. \tag{14}$$

The corresponding step size is h_m which can cause the largest energy change at one integration step.

$$h_m = \sqrt{\frac{6M}{K}} \tag{15}$$

B. Considering a large disturbance

Subjecting to a large disturbance, a nonlinear SMIB model is required and used here. The classical SMIB model about p and q in the Hamiltonian system form is:

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$$\begin{cases} \dot{p} = P_m - P_{\max} \sin q - K_D p \\ \dot{q} = \dfrac{p}{M}, \end{cases} \tag{16}$$

where

$$\begin{cases} M = \dfrac{2H}{\omega_0} \\ q = \delta \\ p = M\dot{q}. \end{cases} \tag{17}$$

where P_m is the mechanical power, P_{max} is the steady-state limit of P_m, K_D is the damping parameter, ω_0 is the nominal frequency and H is the inertia. Now assume that there is no damping in the system which means $K_D = 0$. The total energy can be calculated by

$$E = p^T \frac{1}{2M} p + P_{\max}(\cos\delta_s - \cos q) - P_m(q - \delta_s). \tag{18}$$

where δ_s is the angle of the equilibrium.

Consider a 3rd order approximation of (16) by Taylor Expansion at the equilibrium. There is

$$\begin{cases} \Delta\dot{p} = -K_c \Delta q + \dfrac{K_s}{2}\Delta q^2 + \dfrac{K_c}{6}\Delta q^3 + o(\Delta q^4) \\ \Delta\dot{q} = \dfrac{\Delta p}{M}, \end{cases} \tag{19}$$

where

$$\begin{cases} \Delta p = p - p_s = p - \delta_s \\ \Delta q = q - q_s = q \\ K_c = P_{\max}\cos\delta_s \\ K_s = P_{\max}\sin\delta_s = P_m. \end{cases} \tag{20}$$

For a small neighborhood of the equilibrium, such an approximation is acceptable and can largely simplify the formulae on energy changes for the SMIB system. Thus, the total energy is approximated by

$$\begin{aligned} E &= p^T \frac{1}{2M} p + P_{\max}(\cos\delta_s - \cos q) - P_m(q - \delta_s) \\ &= \Delta p^T \frac{1}{2M}\Delta p - P_m \Delta q \\ &\quad + P_{\max}(\sin\delta_s \Delta q + \frac{1}{2}\Delta q^T \cos\delta_s \Delta q + o(\Delta q^3)) \\ &= \Delta p^T \frac{1}{2M}\Delta p + \Delta q^T \frac{K_c}{2}\Delta q + o(\Delta q^3). \end{aligned} \tag{21}$$

Use (19)-(21) to calculate the change of the total system energy at each iteration with the R-K 4 method:

$$\begin{aligned} \Delta E_{N+1} &= E_{N+1} - E_N \\ &= \frac{\Delta p_{N+1}^2}{2M} + \frac{K_c \Delta q_{N+1}^2}{2} - \frac{\Delta p_N^2}{2M} - \frac{K_c \Delta q_N^2}{2} \\ &= -h^6 K_c^3 (\frac{-K_c h^2 + 8M}{576M^4}) E_N + h\Delta q_N^2 \times \\ &\quad \left\{ \begin{array}{l} 32M^3 h^2 \Delta q_N \Delta p_N (-K_c h^2 + 6M) + \\ 192 K_s M^3 \Delta p_N (-K_c h^2 + 6M) + \\ 2K_s K_c M h \Delta q_N \times \\ (h^6 K_c^3 + 8h^4 M K_c^2 + 24 h^2 K_c M^2 - 288 M^3) \end{array} \right\} \\ &\quad + o(\Delta q_N^4). \end{aligned} \tag{22}$$

The critical value of the integration time step can be solved from (22) by letting $\Delta E_{N+1} = 0$. Note that the resulting critical value varies with the system state and time. To obtain these critical values, an approach is to first use a very small integration time step to generate accurately enough simulation and then to solve critical values by (22) along with the simulation result. Then, similar conclusions on the critical value for the harmonic oscillator can be drawn: if the actual integration time step adopted at a step is less than the critical value at that state, the energy will decrease; otherwise, energy will increase. When the disturbance is small, the resulting critical value matches the critical time step with the harmonic oscillator.

C. Numerical damping of a simulated system

This section adopts the concept of "numerical damping" for the SMIB system as well as the dominant mode of a multi-machine power system. It is defined as the fake portion of the damping observed from numerical simulation if the integration methods cannot preserve the total energy.

Assume $h < h_c$ for the SMIB system. First, consider the corresponding harmonic oscillator with damping. The energy will change with time as described by

$$E = (\frac{p_N^2}{2M} + \frac{K q_N^2}{2}) \times D^N = (\frac{(p_N^e)^2}{2M} + \frac{K(q_N^e)^2}{2}) = E_0 \times D^N \tag{23}$$

where

$$N = t/h$$

$$D = (1 - h^6 K^3 (\frac{-Kh^2 + 8M}{576M^4}))$$

$$\begin{cases} p_N^e = e^{\frac{\ln D}{2h} \times t} p_N \\ q_N^e = e^{\frac{\ln D}{2h} \times t} q_N. \end{cases} \tag{24}$$

In (24), p^e and q^e are equivalent state variables on the dominant oscillation that consider the decreasing energy, and $e^{\ln D/2h}$ is the damping used to illustrate the change of energy caused by the R-K4 method. When the harmonic oscillator system is stable, there is $D < 1$, the error will increase as an exponential function with respect to time t. Since the kinetic energy and potential energy will decrease equally, total additional damping will be

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added to $p(t)$ and $q(t)$ in the same ratio. For a harmonic oscillator system, there is only one mode, whose natural frequency is

$$\omega_n = \sqrt{\frac{K}{M}}. \tag{25}$$

Then, from (23)-(25) the numerical damping, which is the portion of fake damping due to numerical simulation, is given by

$$\zeta_e = -\frac{\ln D}{2h\omega_n} = -\frac{\ln D}{2h} \times \sqrt{\frac{M}{K}} \times 100\%. \tag{26}$$

Assume the calculated damping ratio by, e.g., Prony's method, from the simulated waveform to be ζ_m. Then, the correction damping ratio is

$$\zeta_c = \zeta_m - \zeta_e. \tag{27}$$

For simulations of the SMIB system subject to a small disturbance, equations (11), (12), (26), and (27) can be applied to estimate the numerical damping and find the true damping from simulation results.

III. Case Studies

A. R-K 4 Method on an SMIB system

Assume $\delta_s = 0.524$ rad, $M = 0.1$ s^2/ rad, $P_{\max} = 1$ p. u., $P_m = 0.5$ p. u. Consider a post-disturbance initial state that causes a rotor angle deviation of 0.2 rad. There is no damping in the system. Since this disturbance is not large, higher-order terms in (22) can be ignored to obtain $h_c = 0.9611$ s. Three different time step values are chosen to simulate the system and the energies are compared to the total energy calculated by (21) in Fig.1. From Fig.1, the total energy will decrease if $h < h_c$, increase if $h > h_c$, or oscillate around a constant value if h is around h_c.

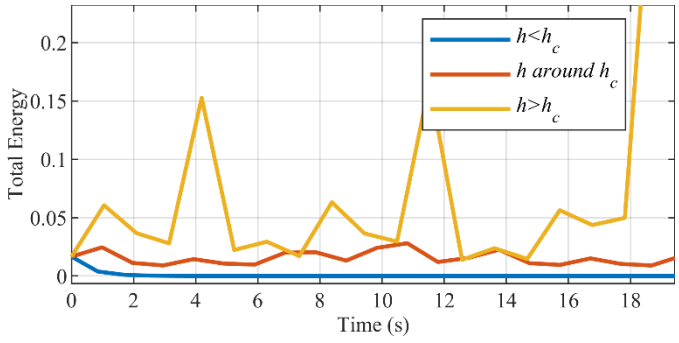

Fig. 1. Total energy for SMIB system

Then, the SMIB system with damping is considered. Assume $K_D = 0.25$ p. u. in (16). The true damping ratio from the linearized model of (16) is 4.2476% at a natural frequency of 0.4684 Hz. For the same disturbance, the R-K 4 method is used to simulate and Prony's method is applied to estimate the damping ratio. With different step sizes, corresponding numerical damping ratios, i.e. the portions of fake damping added by numerical integration, are shown in Fig. 2 and Table I.

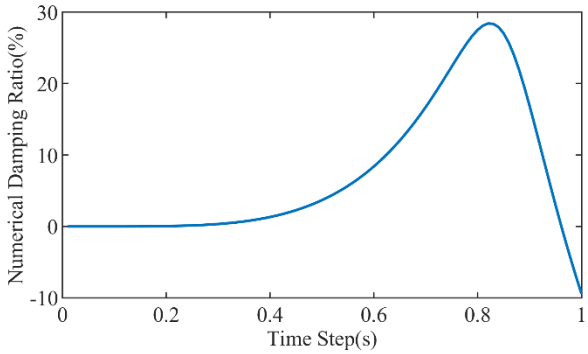

Fig. 2. Numerical damping for the SMIB system

From Fig.2 the numerical damping is highest at h_m=0.8324 s by (15). If a relatively accurate damping ratio is desired when the simulation is numerically stable, the step size should be less than h_m, and the corresponding damping ratios estimated by Prony's method on simulation results are shown in Table I below.

TABLE I. Comparison of Damping Ratio of SMIB System

Step size(s)	Damping Ratio of SMIB System			
	Prony Analysis (%)	Error ε_1 of Prony Analysis (%)	Numerical Damping (%)	Error ε_2 due to numerical damping (%)
0.1	4.2374	0.0102	0.0015	0.0117
0.2	4.2796	0.0320	0.0469	0.0149
0.3	4.5211	0.2735	0.3372	0.0637
0.4	5.4524	1.2048	1.3179	0.1131
0.5	7.8856	3.6380	3.6864	0.0484
0.6	13.0510	8.8034	8.4067	0.3967
0.7	22.5863	18.3387	16.7444	1.5943
0.8	35.5024	31.2548	31.7761	0.5213

Two error indices in Table I are defined by:

$$\begin{cases} \varepsilon_1 = |\zeta_m - \zeta_t| \\ \varepsilon_2 = |\zeta_c - \zeta_t|, \end{cases} \tag{28}$$

where $\zeta_t = 4.2476\%$ is the true damping, ε_1 represents the absolute error of Prony's method, ε_2 represents the absolute error of correction damping in (27). The second column represents damping ratios estimated by Prony's method; the third column gives values of ε_1; the fourth column represents numerical damping ratios calculated by (26), and the last column gives values of ε_2. From Table I, it is obvious that considering the numerical damping ratio can reduce errors introduced by the R-K 4 method, which also verifies inevitable damping errors occur when a large step size is adopted for measurement based mehods.

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B. *R-K 4 Method on a two-area system*

Kundur's two-area system from [12] as shown in Fig.3 is used to test how the system energy can change by the R-K 4 method with the integration time step, which will influence the observed damping of the dominant mode. For instance, a decreasing energy caused by the R-K 4 method presents extra positive damping.

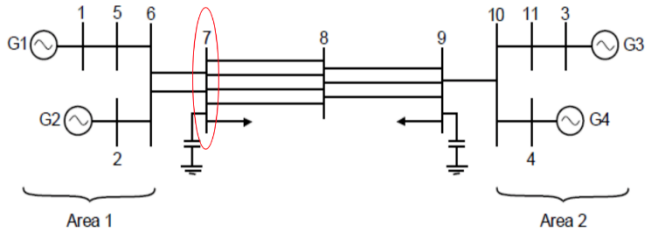

Fig. 3. Two-area system

Consider a temporary 3-phase fault on bus 7 that is cleared after 2 cycles without any tripping line. The relative rotor angles are shown in Fig.4 with a dominant inter-area mode around 0.5 Hz.

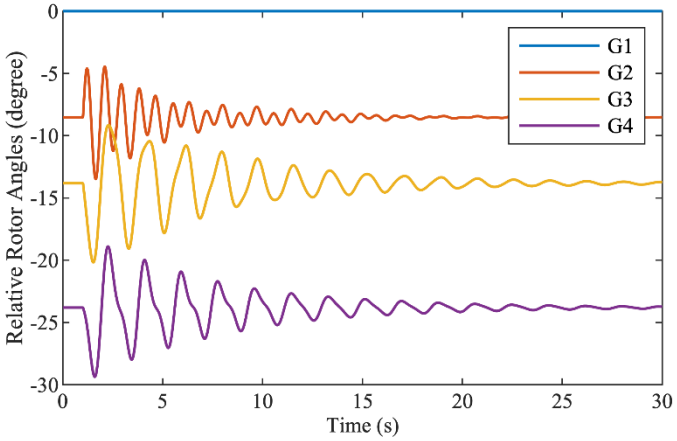

Fig. 4. Relative rotor angles

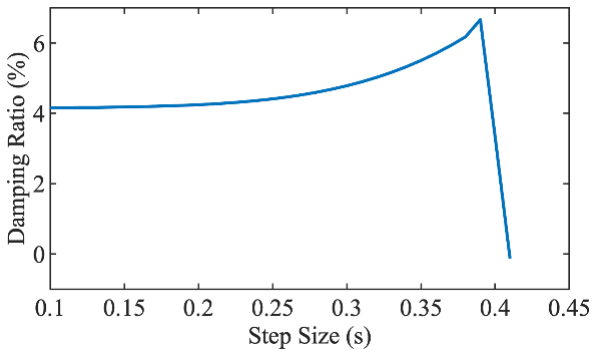

Fig. 5. Damping ratios of the slow mode

For this disturbance, the M and K values of an SIMB equivalent of the two-area system can be found by either model reduction or a modal decomposition based approach. Then, apply (12) to determine the critical time step. Generator 1 is chosen as the reference generator and the simulation result of generator 2 is used for analysis. Several step sizes are adopted to simulate the system. The damping ratios of the inter-area mode are estimated by Prony's method as shown in Fig.5. There is a critical time step h_c at 0.4041 s calculated by (12). It can be verified by Fig.5. When the time step is close to h_c, the change of energy is almost 0. If the time step is less than 0.3 s, the damping ratio will be close to the actual value of 4.1726%.

IV. Conclusion and Future Work

This paper has studied energy preservability with two explicit R-K methods for time-domain simulation. It is discovered that the Euler method is unable to preserve the system energy while the R-K 4 method has a critical time step enabling energy-preserving simulations. The concept of numerical damping has been adopted to evaluate the fake portion of damping told from simulation results if energy is not preserved. The derived formulae and conclusions have been validated on the SMIB system and also tested on a two-area system. Future work will extend the presented approach to multi-machine power systems and apply the formulae to improve accuracy of power system simulation.

References

[1] Y. Liu, K. Sun, R. Yao, B. Wang, "Power System Time Domain Simulation Using a Differential Transformation Method," *IEEE Transactions on Power Systems*, vol. 34, No. 5, pp. 3739-3748, Sep. 2019

[2] B. Park, K. Sun, A. Dimitrovski, Y. Liu, S. Simunovic, "Examination of Semi-Analytical Solution Methods in the Coarse Operator of Parareal Algorithm for Power System Simulation," *IEEE Transactions on Power Systems*, vol. 36, No. 6, pp. 5068-5080, Nov. 2021

[3] B. Wang, N. Duan, K. Sun, "A Time-Power Series Based Semi-Analytical Approach for Power System Simulation," *IEEE Transactions on Power Systems*, vol. 34, No. 2, pp. 841-851, March 2019

[4] N. Duan, K. Sun, "Power System Simulation Using the Multistage Adomian Decomposition Method," *IEEE Transactions on Power Systems*, vol. 32, no. 1, pp. 430-441, Jan. 2017

[5] L. M. Tolbert, F. Wang, K. Tomsovic, K. Sun, J. Wang, Y. Ma, Y. Liu "Reconfigurable Real-Time Power Grid Emulator for Systems with High Penetration of Renewables," *IEEE Open Access Journal of Power and Energy*, vol. 7, pp. 489-500, 2020

[6] F. Li, K. Tomsovic and H. Cui, "A Large-Scale Testbed as a Virtual Power Grid: For Closed-Loop Controls in Research and Testing," in *IEEE Power and Energy Magazine*, vol. 18, no. 2, pp. 60-68, March-April 2020.

[7] H. Liu, K. Zhang, Multi-symplectic Runge–Kutta-type methods for Hamiltonian wave equations, *IMA Journal of Numerical Analysis*, vol 26, no. 2, pp. 252–271, Apr. 2006

[8] K. Bathe, M. M. Irfan Baig, "On a composite implicit time integration procedure for nonlinear dynamics," *Computers & Structures*, vol. 83, no. 31–32, pp. 2513-2524, Dec. 2005.

[9] Chung, J., and Hulbert, G. M. (June 1, 1993). "A Time Integration Algorithm for Structural Dynamics With Improved Numerical Dissipation: The Generalized-α Method." ASME. *J. Appl. Mech*, vol. 60, no. 2, pp. 371–375, Jun. 1993.

[10] C. Fu, J. D. McCalley and J. Tong, "A Numerical Solver Design for Extended-Term Time-Domain Simulation," in *IEEE Transactions on Power Systems*, vol. 28, no. 4, pp. 4926-4935, Nov. 2013.

[11] D. Fabozzi, T. Van Cutsem, "On Angle References in Long-Term Time-Domain Simulations," in *IEEE Transactions on Power Systems*, vol. 26, no. 1, pp. 483-484, Feb. 2011.

[12] P. Kundur, *Power System Stability and Control*. Palo Alto, CA: McGraw-Hill, 1993, pp. 813–814.

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