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Necessary and sufficient conditions for quadratic stabilizability of switched systems on non-uniform time domains☆

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A B S T R A C T

In this paper, we consider the quadratic stabilizability via state feedback for a particular class of switched systems that evolve on a non-uniform time domain by introducing time scales theory. The system considered switches between a continuous-time subsystem with variable lengths and a discrete-time subsystem with variable discrete step sizes. Necessary and sufficient conditions are derived to guarantee the quadratic stability of this class of switched systems via a switching state feedback law based on the existence of a common positive definite matrix satisfying the quadratic stabilizability condition by considering that the two subsystems are unstable. By state feedback, we mean that the switching among subsystems depends on the system states. Current results for this kind of state switching feedback control are derived only for switched systems evolving on a continuous time domain or a discrete time domain with fixed step's size. These results are not applicable for the particular class of switched systems where there is a mixing between the continuous and discrete dynamics. This motivates the derivation of a new and more general state feedback control law for switched systems in this work. A numerical example illustrating the results is presented.

1. Introduction

Switching among different system structures is an essential feature of many engineering control applications since there is a large number of systems controlled by different controller structures or control-laws. Moreover, many practical systems are inherently multi-modal in the sense that several dynamical subsystems are required to describe their behavior that may depend on various environmental factors, where the methods of intelligent control design are based on the idea of switching between different controllers [1–5]. By a switched system, we mean a hybrid dynamical system that is composed of a family of continuous-time and/or discrete-time subsystems and a rule orchestrating the switching between them [2,6]. The stability of the closed-loop controlled system depends in general on the switching strategy. Switching between stable subsystems does not necessarily imply a stable closed-loop behavior [2,7]. In contrast, by designing a proper switching strategy unstable systems can be stabilized [1,8–10].

Switched systems have been studied from various viewpoints [8]. One viewpoint is that the switching signal is an exogenous variable, and then the problem is to give conditions to guarantee that the switched system has the desired

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performance (stability, a certain disturbance attenuation level, etc.) [11,12]. Another viewpoint, which is of interest here, is that the switching signal may be used for control purposes, such that if the switching rule is properly chosen, the switched system is stabilized. There are a few existing results concerning the quadratic stabilization of switched systems that are composed of unstable linear time-invariant subsystems. In [1,11] it has been shown that the existence of a stable convex combination of the subsystem state matrices implies the existence of a state-dependent switching rule that stabilizes the switched system. It has been proved in [8] that when the number of subsystems is two, the existence of a stable convex combination of the subsystem state matrices is necessary and sufficient for quadratic stabilizability of the switched system by state-dependent switching. In [9,13] this problem has been studied for discrete-time switched systems. Note that, this concept of quadratic stabilization has been studied for continuous-time switched systems and discrete-time switched systems separately by providing conditions to guarantee the quadratic stability of the switched system via state based on the existence of a common positive definite matrix satisfying the quadratic stabilizability condition.

In this paper, we are interested in generalizing the quadratic stabilizability concept via a switching state feedback law to a special class of switched systems evolving on a non uniform time domain (not continuous everywhere, nor discrete everywhere with fixed step's size). The system considered switches between a continuous-time subsystem (on some intervals with variable lengths) and a discrete-time subsystem with variable discrete step sizes. This kind of switched systems is different from traditional hybrid systems [6], since in the latter, the discrete event (or the jump) is assumed instantaneous and the relationship between the state at the jump time and after the jump is determined by an algebraic map. In this work we are considering that the discrete event (the jump) is not instantaneous and to reach the state after the jump, the system will take some time, denoted $\mu(t)$, that is time varying, such that, the relationship between the state at the jump time and after the jump is determined by a discrete-dynamic with non-fixed discrete steps. In another word, in this framework, we are generating the jump dynamically and not algebraically. Since the time domain of this class of switched systems is neither continuous nor discrete, the time scales theory [14] is introduced to study the quadratic stabilizability of this special class of switched systems. This theory serves to unify continuous and discrete analysis and generalizes the study to dynamical systems on any non-uniform time domain which can be discrete with variable discrete step sizes or a mixture between continuous and discrete parts. This class can describe a wide range of physical and engineering systems. For instance, cooperative control over networks [15,16], such that the controller assumes that local information is exchanged over some disconnected time intervals due to communication obstacles or sensor failure. This concept has been introduced in the problem of consensus for linear multi-agent systems and the observer based leader–follower bipartite consensus with intermittent information transmission in [17,18], and using the stochastic approach on time scales in [19,20]. In [21], the problem of wide area damping controller in power systems has been converted to a switched system between continuous-time and discrete-time subsystems, and the maximum allowable delay that guarantees the stability and performance of the system, has been derived by introducing time scales theory. In [22], this class of switched systems has been introduced for modeling intermittent hormone therapy for prostate cancer.

The stability of this special class of switched system has been studied by introducing time scales theory in several works. In [23,24], dwell time conditions have been derived for the stabilization of this particular class of switched systems by considering that unstable modes may exist and by introducing multiple Lyapunov functions. The common Lyapunov function has been introduced in [25], and some stability conditions have been derived by using the solution of the system [26]. To the best of our knowledge, the problem of the quadratic stabilizability of this class of switched systems via a state feedback switching rule by considering that both subsystems are unstable has not been studied. The motivation is to investigate necessary and sufficient conditions for quadratic stabilizability by introducing time scales Lyapunov functions. The control objective is to choose which mode has to be activated, according to some conditions, to stabilize the overall system. The derived quadratic stabilizability condition is expressed as a matrix inequality with a positive scalar and a common positive definite matrix. Note that, the novelty of this work is that a particular class of switched systems is considered where there is a mixing between continuous and discrete dynamics. Current results for this kind of state switching feedback control are derived only for switched systems evolving on a continuous time domain or a discrete time domain with a fixed step's size. These results are not applicable for the class of switched systems considered motivating the derivation of the results presented here. In particular, it is shown that the new derived results generalize both the continuous case and the discrete case. One challenging problem, is that even if the discrete subsystem is time invariant the derivative of the quadratic Lyapunov function along the trajectories of this system depends on the step's size which is variable in time and makes this derivative variable in time too. A numerical example is presented to show the effectiveness of the approach.

2. Preliminaries on time scales theory

In this section, we recall some basics on time scales theory which are necessary for the comprehension of the result presented in this paper (see [14] for more details).

A *time scale* $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$. We define the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t \in \mathbb{T}\}$ which gives the time that comes just after t. The *backward jump operator* is defined by $\rho : \mathbb{T} \to \mathbb{T}$ by $\rho(t) = \sup\{s \in \mathbb{T} : s < t \in \mathbb{T}\}$ which gives the time that comes just before t. The mapping $\mu : \mathbb{T} \to \mathbb{R}^+$, called the *graininess function*, is defined by $\mu(t) = \sigma(t) - t$ which measure the distance between two consecutive times. For $\mathbb{T} = \mathbb{R}$, we have $\rho(t) = t = \sigma(t)$ and $\mu(t) = 0$. For $\mathbb{T} = h\mathbb{Z}$, we have $\rho(t) = t - h$, $\sigma(t) = t + h$ and $\mu(t) = h$. The set $\mathbb{T}^\kappa$ is defined as follows: if $\mathbb{T}$ has a left-scattered maximum m (i.e, $\rho(m) < m$), then $\mathbb{T}^\kappa = \mathbb{T} - \{m\}$; otherwise $\mathbb{T}^\kappa = \mathbb{T}$. Since $\mathbb{T}$ can be any non-uniform time domain, we need to define the derivative on this particular set. Let $f : \mathbb{T} \to \mathbb{R}$, the Δ-derivative of f at $t \in \mathbb{T}^\kappa$ is defined by

$$f^\Delta(t) = \lim_{s\to t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s}. \tag{1}$$

One can notice that this derivative unify the continuous and the discrete one. If $\mathbb{T} = \mathbb{R}$, then $f^\Delta(t) = \dot{f}(t)$, which is the euclidean derivative of f, and if $\mathbb{T} = h\mathbb{Z}$, then $f^\Delta(t) = (f(t+h) - f(t))/h$. For $h = 1$, $f^\Delta(t) = f(t+1) - f(t) = \Delta f(t)$ which is the difference operator. A function $p : \mathbb{T} \to \mathbb{R}$ is *regressive* if $1 + \mu(t)p(t) \neq 0$, $\forall t \in \mathbb{T}^\kappa$. We denote the set of regressive functions by $\mathcal{R}$, and by $\mathcal{R}^+$ if they satisfy $1 + \mu(t)p(t) > 0$, $\forall t \in \mathbb{T}^\kappa$ (i.e *positively regressive functions*). Similarly, a matrix $A : \mathbb{T} \to \mathbb{R}^n$ is said to be *regressive* if $(I + \mu(t)A)$ is invertible or equivalently, if and only if all its eigenvalues are regressive.

Let the first-order dynamical system on $\mathbb{T}$ defined by

$$x^\Delta(t) = Ax(t), \quad x(t_0) = x_0, \quad t_0 \in \mathbb{T}, \quad A \in \mathbb{R}^{n\times n}. \tag{2}$$

Theorem 1 (*[14]*)**.** *For $A \in \mathcal{R}$, the unique solution of (2) is given by the generalized exponential function $e_A(t, t_0)x_0$.*

The *generalized exponential function* of $p : \mathbb{T} \to \mathbb{R}$, with $p \in \mathcal{R}$, on an arbitrary $\mathbb{T}$ is expressed, for $t, s \in \mathbb{T}$, by

$$e_p(t,s) = \begin{cases} \exp\left(\int_s^t \frac{\log(1+\mu(\tau)p(\tau))}{\mu(\tau)} \Delta\tau\right), & \mu(t) \neq 0 \\[2ex] \exp\left(\int_s^t p(\tau)\Delta\tau\right), & \mu(t) = 0. \end{cases} \tag{3}$$

where log is the principal logarithm function and the Δ-integral is used [27]. This exponential function generalizes the solution of (2) on any non-uniform time domain. For example, for $\mathbb{T} = h\mathbb{Z}$, we have $e_p(t,s) = \prod_{\tau=s}^{t-h}(1 + hp(\tau))$. Note that, the regressivity of p is needed for the generalized exponential to be well defined. If $p \in \mathcal{R}^+$ (positively regressive), the generalized exponential function is always positive. If p is not positively regressive (i.e; $\exists t \in \mathbb{T}^\kappa : 1 + \mu(t)p < 0$), the generalized exponential function can be negative (see [14]). If $\mathbb{T}^\kappa = \mathbb{R}$, the positive regressivity is always satisfied since $\mu(t) = 0$ and $1 + \mu(t)p = 1 > 0$.

The stability and asymptotic stability definitions on $\mathbb{T}$ are similar to the standard stability concepts. The exponential stability is achieved by some modification [28,29]. The system (2) is exponentially stable on $\mathbb{T}$, if there exists a constant $\beta \geq 1$ and a negative constant $\lambda \in \mathcal{R}^+$, such that the corresponding solution satisfies

$$\|x(t)\| \leq \beta\|x_0\|e_\lambda(t, t_0), \quad \forall t, t_0 \in \mathbb{T}. \tag{4}$$

This characterization is a generalization of the definition of exponential stability for dynamical systems defined on $\mathbb{R}$ or $h\mathbb{Z}$. More specifically, the condition that $\lambda < 0$ and $\lambda \in \mathcal{R}^+$ is reduced to $\lambda < 0$ for $\mathbb{T} = \mathbb{R}$, and to $0 < 1 + \mu(t)\lambda < 1$, $\forall t \in \mathbb{T}$ any discrete time scale. Since, it is difficult to determine the region of exponential stability of (2) on an arbitrary $\mathbb{T}$ [29], the notion of Hilger circle $\mathcal{H}_{\mu(t)}$ is introduced:

$$\mathcal{H}_{\mu(t)} := \left\{ z \in \mathbb{C} : \ |1 + z\mu(t)| < 1, \ z \neq -\frac{1}{\mu(t)} \right\}, \text{ where } \mathbb{C} \text{ is the set of complex numbers.}$$

For $0 \leq \mu(t) \leq \mu_{\max} = \sup_{t\in\mathbb{T}} \mu(t)$, there exists a Hilger circle $\mathcal{H}_{\min}$ associated to $\mu_{\max}$, such that if all the eigenvalues of A are in $\mathcal{H}_{\min}$, then the system (2) is exponentially stable (see [28]). The Lyapunov stability is also generalized to dynamical systems evolving on an arbitrarily $\mathbb{T}$ in [30–32].

Definition 1 (*[30]*)**.** $V : \mathbb{R}^n \to \mathbb{R}$ is called a time scale Lyapunov function (TSLF) for the system (2) if, $\forall t \in \mathbb{T}$,

(i) $V(x(t)) \geq 0$, with equality if and only if $x(t) = 0$,
(ii) $V^\Delta(x(t)) \leq 0$ along the trajectories of (2).

Theorem 2 (*[30]*)**.** *Consider the system (2). If there exists an associated TSLF V, then the equilibrium is stable. Furthermore, if $V^\Delta(x(t)) < 0$, when $x(t) \neq 0$, then the equilibrium is asymptotically stable.*

A standard approach is to seek an associated quadratic TSLF $V(x(t)) = x^T(t)Px(t)$, with $P = P^T > 0$. The Δ-derivative of V along the trajectories of system (2), on an arbitrary $\mathbb{T}$, is given by [30]

$$V^\Delta(x(t)) = x^T(t)(A^TP + PA + \mu(t)A^TPA)x(t). \tag{5}$$

Therefore, we seek a solution P to the following time scale algebraic Lyapunov equation (TSALE)

$$A^TP + PA + \mu(t)A^TPA = -Q, \quad Q = Q^T > 0. \tag{6}$$

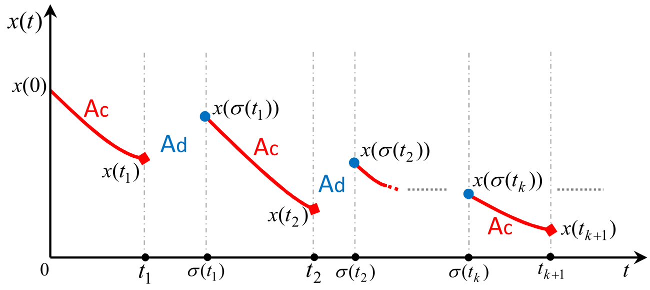

Fig. 1. Switched system on $\mathbb{T} = \mathbb{P}_{\{\sigma(t_k),t_{k+1}\}}$.

If $\mathbb{T} = \mathbb{R}$, $\mu(t) = 0$, then (6) reduces to the standard algebraic equation for continuous-time dynamic. If $\mathbb{T} = \mathbb{Z}$, $\mu(t) = 1$, and (6) reduces to

$$A^T P + PA + A^T PA = (I + A)^T P(I + A) - P = -Q, \tag{7}$$

which coincides with the algebraic Lyapunov equation of the standard recursive equation $x(t+1) = (I+A)x(t)$. The matrix P which solves (6) is given by (see [31])

$$P = \int_{t_0}^{t} e_{A^T}(s, t_0)Q(t)e_A(s, t_0)\Delta s, \quad t \geq t_0.$$

Notice that (6) unifies the TSALE on an arbitrary $\mathbb{T}$ and it is generally time-varying because of the time-varying $\mu(t)$.

Remark 1. If the TSALE (6) is satisfied for $\mu_{\max}$, then it is satisfied for all $\mu(t) \leq \mu_{\max}$, since $A^T P + PA + \mu(t)A^T PA \leq A^T P + PA + \mu_{\max}A^T PA = -Q,\ \forall t \in \mathbb{T}$, which can simplify the computation of P (see [32]).

3. Problem statement

Time scales theory is introduced to study the quadratic stabilizability of a special class of switched systems via state feedback, where the system switches between a continuous-time linear subsystem and a discrete-time linear subsystem during a certain period of time. Consider the particular time scale

$$\mathbb{T} = \mathbb{P}_{\{\sigma(t_k),t_{k+1}\}} = \cup_{k=0}^{\infty}[\sigma(t_k), t_{k+1}] \tag{8}$$

where $\sigma(.)$ is the forward jump operator, such that, $\sigma(t_0) = t_0$ and the graininess function $\mu(t_k) = \sigma(t_k) - t_k, \forall k \in \mathbb{N}^*$ ($\mathbb{N}^* = \mathbb{N}\setminus\{0\}$). Note that, $\mu(t)$ should be bounded such that $0 < \mu(t) \leq \mu_{\max},\ \forall t \in \mathbb{T}$. Let $\{A_c, A_d\}$ be a set of two constant regressive matrices in $\mathbb{R}^{n\times n}$. The switched system on $\mathbb{T} = \mathbb{P}_{\{\sigma(t_k),t_{k+1}\}}$ is determined by

$$x^{\Delta}(t) = \begin{cases} A_c x(t); & \text{for} \quad t \in \cup_{k=0}^{\infty}[\sigma(t_k), t_{k+1}[\\ A_d x(t); & \text{for} \qquad t \in \cup_{k=0}^{\infty}\{t_{k+1}\}. \end{cases} \tag{9}$$

The first equation of (9) describes the continuous-time linear dynamic and the second is the discrete-time linear dynamic, during a variable period of time $\mu(t_k) = \sigma(t_k) - t_k$, for $k \in \mathbb{N}^*$, (which may correspond to the time needed for the state jump) Fig. 1. The sequence $\{t_0, t_1, \sigma(t_1), t_2, \sigma(t_2), \ldots\}$ is a monotonically increasing sequence of switching times without finite accumulation points.

The switched system (9) describes the time evolution of each subsystem and can be written as

$$x^{\Delta}(t) = A_{\kappa(x,t)}x. \tag{10}$$

where $\kappa(x, t) : \mathbb{R}^n \times \mathbb{R} \to \{c, d\}$ is a piecewise constant function which represents the switching rule. The main objective is to determine a state control feedback switching rule that quadratically stabilize the trajectories of (10) by proper choice of $\kappa(x, t)$ (i.e, select the mode A_c or A_d to be activated according to some state conditions). By state feedback, we mean that the switching among subsystems depends on the system states.

Definition 2 ([8]). The switched system (10) is quadratically stabilizable via state feedback if and only if there exists a positive definite function $V(x) = x^T Px$, a positive number $\varepsilon > 0$ and a switching rule $\kappa(x, t)$ such that

$$V^{\Delta}(x(t)) < -\varepsilon x^T(t)x(t), \quad \text{for all trajectories } x(t) \text{ of (10)}.$$

Note that, the Δ-derivative of $V(x)$ along the trajectories of the switched system (9) is given by

$$V^{\Delta}(x(t)) = \begin{cases} x^T(t)(A_c^T P + PA_c)x(t), & t \in \cup_{k=0}^{\infty}[\sigma(t_k), t_{k+1}[\\ x^T(t)(A_d^T P + PA_d + \mu(t)A_d^T PA_d)x(t), & t \in \cup_{k=0}^{\infty}\{t_{k+1}\}. \end{cases} \tag{11}$$

To prove the main result, we shall need the following Lemma.

Lemma 1 (*[33]. [2] (S-procedure)*). *Let $T_0, T_1 \in \mathbb{R}^{n\times n}$ be two symmetric matrices. Consider the following conditions:*

$$x^T T_0 x > 0 \ \text{ for all } x \neq 0 \ \text{ such that } \ x^T T_1 x \geq 0, \tag{12}$$

and

$$\exists \tau_1 \geq 0 \ \text{ such that } \ T_0 - \tau_1 T_1 > 0. \tag{13}$$

Then condition (13) always implies condition (12). If there exists $x_0 \neq 0$ such that $x_0^T T_1 x_0 > 0$, then condition (12) implies condition (13).

Next, we will derive a necessary and sufficient condition for the state feedback quadratic stabilizability of the class of the switched system (10) evolving on $\mathbb{T} = \cup_{k=0}^{\infty}[\sigma(t_k), t_{k+1}]$ as (9) .

4. Main results

In this section, we generalize the results for the quadratic stabilizability derived for the continuous-time and discrete-time switched systems [1,8,9,13], to the class of switched systems (10). First, consider the following assumption:

Assumption 1. Assume that A_c and A_d are both unstable, which means that there does not exist a matrix $P_1 = P_1^T > 0$ satisfying the inequalities $x^T(t)(A_c^T P_1 + P_1 A_c)x(t) < 0$, for all the trajectories $x(t)$ of the continuous-time subsystem in (10), nor there exists $P_2 = P_2^T > 0$ satisfying $x^T(t)(A_d^T P_2 + P_2 A_d + \mu(t) A_d^T P_2 A_d)x(t) < 0$, for all the trajectories $x(t)$ of the discrete-time subsystem in (10).

Theorem 3. *Suppose that Assumption 1 holds. The switched system (10) is quadratically stabilizable via state feedback if and only if, there exists $P = P^T > 0$, $0 < \alpha < 1$ and $\varepsilon > 0$, such that the following inequality is satisfied*

$$[\alpha A_c + (1-\alpha)A_d]^T P + P[\alpha A_c + (1-\alpha)A_d] + (1-\alpha)\mu_{\max} A_d^T P A_d < -\varepsilon I. \tag{14}$$

Proof (*Sufficiency*). Suppose that there exists $P = P^T > 0$, $0 < \alpha < 1$ and $\varepsilon > 0$, such that condition (14) holds. Then for any $x(t) \neq 0$, we have

$$x^T(t)[(\alpha A_c + (1-\alpha)A_d)^T P + P(\alpha A_c + (1-\alpha)A_d) + (1-\alpha)\mu_{\max} A_d^T P A_d]x(t) < -\varepsilon x^T(t)x(t), \tag{15}$$

which is equivalent to

$$\alpha x^T(t)[A_c^T P + PA_c]x(t) + (1-\alpha)x^T(t)[A_d^T P + PA_d + \mu_{\max} A_d^T P A_d]x(t) < -\varepsilon x^T(t)x(t). \tag{16}$$

Inequality (16) can be written as

$$\alpha x^T(t)[A_c^T P + PA_c + \varepsilon I]x(t) + (1-\alpha)x^T(t)[A_d^T P + PA_d + \mu_{\max} A_d^T P A_d + \varepsilon I]x(t) < 0. \tag{17}$$

According to Assumption 1, we conclude from the above inequality that either

$$x^T(t)[A_c^T P + PA_c]x(t) < -\varepsilon x^T(t)x(t) \tag{18}$$

or

$$x^T(t)[A_d^T P + PA_d + \mu_{\max} A_d^T P A_d]x(t) < -\varepsilon x^T(t)x(t) \tag{19}$$

is true. For example when (18) is not true with $x^T(t)[A_c^T P + PA_c]x(t) \geq -\varepsilon x^T(t)x(t)$, the inequality (19) has to be satisfied from (17).

Note that, if (19) is satisfied, we have

$$x^T(t)[A_d^T P + PA_d + \mu_{\max} A_d^T P A_d + \mu(t) A_d^T P A_d - \mu(t) A_d^T P A_d]x(t) < -\varepsilon x^T(t)x(t). \tag{20}$$

Since $\mu(t) \leq \mu_{\max}$, $\forall t \in \cup_{k=0}^{\infty}\{t_{k+1}\}$, we get $(\mu(t) - \mu_{\max})A_d^T P A_d < 0$, and (20) implies that

$$x^T(t)[A_d^T P + PA_d + \mu(t) A_d^T P A_d]x(t) \leq (\mu(t) - \mu_{\max})x^T(t)(A_d^T P A_d)x(t) - \varepsilon x^T x \leq -\varepsilon x^T x, \quad \text{for all } \ \mu(t) \leq \mu_{\max}. \tag{21}$$

Now we define the switching strategy as

$$\kappa(x, t) = \arg\min\{x^T(t)(A_c^T P + PA_c)x(t), \ x^T(t)(A_d^T P + PA_d + \mu_{\max} A_d^T P A_d)x(t)\}. \tag{22}$$

This switching rule means that, when $x^T(t)(A_c^T P + PA_c)x(t) < -\varepsilon x^T(t)x(t)$ we activate the continuous-time subsystem until when this inequality will not become true, and at that time the inequality $x^T(t)[A_d^T P + PA_d + \mu_{\max} A_d^T P A_d]x(t) < -\varepsilon x^T(t)x(t)$ will be satisfied and the discrete-time subsystem will be activated. When these two inequalities are satisfied simultaneously, the system corresponding to the inequality which takes the minimum values is activated until when one of them is violated.

Let the quadratic function $V(x) = x^T Px$. The Δ-derivative of V along the trajectories of the switched system (9) is given by (11). Using this fact and the above switching rule, we conclude that

$$V^{\Delta}(x(t)) < -\varepsilon x^T(t)x(t), \tag{23}$$

is true for all trajectories $x(t)$ of (10), and thus the switched system is quadratically stable.

(Necessity) Assume that the switched system (10) is quadratically stabilizable with the quadratic function $V(x) = x^T Px$, $P = P^T > 0$, such that, there exists $\varepsilon > 0$ satisfying that

$$V^{\Delta}(x(t)) < -\varepsilon x^T(t)x(t), \quad \text{with} \quad V^{\Delta}(x(t)) = \begin{cases} x^T(t)(A_c^T P + PA_c)x(t), & t \in \cup_{k=0}^{\infty}[\sigma(t_k), t_{k+1}[\\ x^T(t)(A_d^T P + PA_d + \mu(t)A_d^T PA_d)x(t), & t \in \cup_{k=0}^{\infty}\{t_{k+1}\}. \end{cases} \tag{24}$$

Since A_c and A_d are both unstable, so the inequality $V^{\Delta}(x(t)) < -\varepsilon x^T(t)x(t)$ is satisfied if

$$x^T(t)[A_c^T P + PA_c]x(t) < -\varepsilon x^T(t)x(t) \quad \text{whenever} \quad x^T(t)[A_d^T P + PA_d + \mu(t)A_d^T PA_d]x(t) \geq -\varepsilon x^T(t)x(t) \tag{25}$$

and

$$x^T(t)[A_d^T P + PA_d + \mu(t)A_d^T PA_d]x(t) < -\varepsilon x^T(t)x(t) \quad \text{whenever} \quad x^T(t)[A_c^T P + PA_c]x(t) \geq -\varepsilon x^T(t)x(t). \tag{26}$$

Suppose that there exists $x(t)$ satisfying (25). Then we have

$$-x^T(t)[A_c^T P + PA_c + \varepsilon I]x(t) > 0 \quad \text{whenever} \quad x^T(t)[A_d^T P + PA_d + \mu(t)A_d^T PA_d + \varepsilon I]x(t) \geq 0, \quad \text{for } x(t) \neq 0. \tag{27}$$

According to Lemma 1, condition (27) holds if and only if, there exists $s > 0$, such that

$$(-A_c^T P - PA_c - \varepsilon I) - s(A_d^T P + PA_d + \mu(t)A_d^T PA_d + \varepsilon I) > 0,$$

which implies that

$$(A_c^T + sA_d^T)P + P(A_c + sA_d) + s\mu(t)A_d^T PA_d + \varepsilon(1+s)I < 0.$$

This inequality is equivalent to

$$\left(\left(\frac{1}{s+1}\right)A_c^T + \left(\frac{s}{s+1}\right)A_d^T\right)P + P\left(\left(\frac{1}{s+1}\right)A_c + \left(\frac{s}{s+1}\right)A_d\right) + \left(\frac{s}{s+1}\right)\mu(t)A_d^T PA_d < -\varepsilon I. \tag{28}$$

Let $\alpha = \frac{1}{s+1}$, so (28) becomes

$$(\alpha A_c + (1-\alpha)A_d)^T P + P(\alpha A_c + (1-\alpha)A_d) + (1-\alpha)\mu(t)A_d^T PA_d < -\varepsilon I, \quad \text{with} \quad 0 < \alpha < 1, \tag{29}$$

which concludes the proof. ■

Remark 2. From the Δ-derivative of V along the trajectories of the discrete-time subsystem in (11), we can remark that the stability of the discrete-time subsystem depends on the eigenvalues of A_d and also on $\mu(t)$ at each instant $t \in \cup_{k=0}^{\infty}\{t_{k+1}\}$. More generally, the discrete-time subsystem is Hilger stable, if all the eigenvalues λ_d^j of A_d lie strictly inside the Hilger circle $\mathcal{H}_{\min}$. Which means that

$$|1 + \mu(t)\lambda_d^j| < 1, \ \forall 1 \leq j \leq n, \ \forall t \in \cup_{k=0}^{\infty}\{t_{k+1}\}. \tag{30}$$

Moreover, the discrete-time subsystem is unstable, if there exists at least one eigenvalue λ_d^j of A_d such that

$$|1 + \mu(t)\lambda_d^j| > 1, \ \forall t \in \cup_{k=0}^{\infty}\{t_{k+1}\} \text{ (i.e; } \exists\lambda_d^j \text{ such that } \Re(\lambda_d^j) > 0 \text{ or } \exists\lambda_d^j \text{ such that } \mu(t) > \frac{-2\Re(\lambda_d^j)}{|\lambda_d^j|^2}, \ \forall t \in \cup_{k=0}^{\infty}\{t_{k+1}\})$$

where $\Re(\lambda_d^j)$ denotes the real part of the eigenvalue λ_d^j. [23].

Remark 3. Suppose that we are switching between two continuous-time subsystems, so there will be no gaps between the two time instants t_k and $\sigma(t_k)$, and the switched system (9) becomes

$$x^{\Delta}(t) = \begin{cases} A_{c1}x(t); & \text{for} \quad t \in \cup_{k=0}^{\infty}[\sigma(t_k), t_{k+1}[\\ A_{c2}x(t); & \text{for} \quad t \in \cup_{k=0}^{\infty}[t_{k+1}, \sigma(t_{k+1})[. \end{cases} \tag{31}$$

In this case the Δ-derivative of V along the trajectories of the second subsystem in (31) becomes

$$V^{\Delta}(x(t)) = x^T(t)(A_{c2}^T P + PA_{c2})x(t), \quad t \in \cup_{k=0}^{\infty}[t_{k+1}, \sigma(t_{k+1})[.$$

Which means that there is no gaps (according to (5)), and the condition (14) becomes

$$[\alpha A_c + (1-\alpha)A_d]^T P + P[\alpha A_c + (1-\alpha)A_d] < -\varepsilon I,$$

which coincide with the result for continuous switched systems [8,9].

If we are switching between two discrete-time subsystems, which means that there are gaps between $\sigma(t_k)$ and t_{k+1}, such that $t_{k+1} - \sigma(t_k) = \tau(t_k)$, the switched system (9) becomes

$$x^{\Delta}(t) = \begin{cases} A_{d1}x(t); & \text{for} \quad t \in \cup_{k=0}^{\infty}\{\sigma(t_k)\} \\ A_{d2}x(t); & \text{for} \quad t \in \cup_{k=0}^{\infty}\{t_{k+1}\}. \end{cases} \tag{32}$$

The Δ-derivative of V along the trajectories of the first subsystem in (32) becomes

$$V^{\Delta}(x(t)) = x^T(t)(A_{d1}^T P + PA_{d1} + \tau(t)A_{d1}^T PA_{d1})x(t), \quad t \in \cup_{k=0}^{\infty}\{\sigma(t_k)\}$$

and the condition (14) will be

$$[\alpha A_{d1} + (1-\alpha)A_{d2}]^T P + P[\alpha A_{d1} + (1-\alpha)A_{d2}] + \alpha\tau_{\max}A_{d1}^T PA_{d1} + (1-\alpha)\mu_{\max}A_{d2}^T PA_{d2} < -\varepsilon I.$$

Or equivalently,

$$\alpha[(I + \tau_{\max}A_{d1})^T P(I + \tau_{\max}A_{d1}) - P] + (1-\alpha)[(I + \mu_{\max}A_{d2})^T P(I + \mu_{\max}A_{d2}) - P] < -\varepsilon I,$$

which generalizes the result of the quadratic stabilizability for discrete switched systems given in [9,13], to a switched systems evolving on a discrete-time domain with variable discrete step sizes.

Note that, if we consider a switched system between two discrete dynamics, described by A_{d1} and A_{d2}, on $\mathbb{T} = \mathbb{Z}$ such that $x^{\Delta}(t) = A_{d\{1,2\}}x(t)$. Using the definition of the Δ-derivative, we get

$$x^{\Delta}(t) = \frac{x(\sigma(t)) - x(t)}{\sigma(t) - t} = A_{d\{1,2\}}x(t) \Rightarrow \frac{x(t+1) - x(t)}{t+1-t} = A_{d\{1,2\}}x(t) \Rightarrow x(t+1) - x(t) = A_{d\{1,2\}}x(t)$$
$$\Rightarrow x(t+1) = (I + A_{d\{1,2\}})x(t).$$

By applying the result in Ref. [13], the condition of stability becomes

$$\alpha[(I + A_{d1})P(I + A_{d1}) - P] + (1-\alpha)[(I + A_{d2})P(I + A_{d2}) - P] < 0$$

which coincides with the last inequality given in Remark 3 by taking the discrete steps $\tau_{max} = 1$ and $\mu_{\max} = 1$.

Remark 4. By fixing $\mu_{\max}$ such that A_d is unstable, the condition (14) is a bilinear matrix inequality (BMI) according to both variables P and α. The Eq. (14) is equivalent to

$$[A_c^T P + PA_c] + \left(\frac{1-\alpha}{\alpha}\right)[A_d^T P + PA_d + \mu_{\max}A_d^T PA_d] < -\varepsilon I, \tag{33}$$

The BMI (33) can be solved by fixing $0 < \alpha < 1$, choosing any $\varepsilon = \varepsilon_0$ large enough, then we change ε gradually such as (33) is feasible, and we get the matrix P. Note that to get the range of feasibility of α satisfying

$$[A_c^T P + PA_c] + \left(\frac{1-\alpha}{\alpha}\right)[A_d^T P + PA_d + \mu_{\max}A_d^T PA_d] < 0, \tag{34}$$

we can rewrite the above inequality as follows: We have

$$A_d^T P + PA_d + \mu_{\max}A_d^T PA_d = \frac{1}{\mu_{\max}}[(I + \mu_{\max}A_d^T)P(I + \mu_{\max}A_d) - P]. \tag{35}$$

By substituting (35) into (34), we get the following inequality

$$[A_c^T P + PA_c] + \gamma[(I + \mu_{\max}A_d^T)P(I + \mu_{\max}A_d) - P] < 0. \tag{36}$$

with $\gamma = \left(\frac{1-\alpha}{\mu_{\max}\alpha}\right)$. Using the Schur complement, (36) is equivalent to

$$\begin{bmatrix} A_c^T P + PA_c - \gamma P & -(I + \mu_{\max}A_d^T)P \\ -P(I + \mu_{\max}A_d) & -\frac{1}{\gamma}P \end{bmatrix} < 0. \tag{37}$$

This BMI is a feasibility problem. To find a feasible γ, we can decompose the above matrix as follows:

$$\begin{aligned} \begin{bmatrix} A_c^T P + PA_c - \gamma P & -(I + \mu_{\max}A_d^T)P \\ -P(I + \mu_{\max}A_d) & -\frac{1}{\gamma}P \end{bmatrix} &= \begin{bmatrix} (A_c^T - \frac{\gamma}{2}I) & -(I + \mu_{\max}A_d^T) \\ 0 & -\frac{1}{2\gamma}I \end{bmatrix} \begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix} \\ &+ \begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix} \begin{bmatrix} (A_c - \frac{\gamma}{2}I) & 0 \\ -(I + \mu_{\max}A_d) & -\frac{1}{2\gamma}I \end{bmatrix}. \end{aligned} \tag{38}$$

So (37) is equivalent to the inequality

$$\begin{bmatrix} (A_c^T - \frac{\gamma}{2}I) & -(I + \mu_{\max}A_d^T) \\ 0 & -\frac{1}{2\gamma}I \end{bmatrix} \begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix} + \begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix} \begin{bmatrix} (A_c - \frac{\gamma}{2}I) & 0 \\ -(I + \mu_{\max}A_d) & -\frac{1}{2\gamma}I \end{bmatrix} < 0. \tag{39}$$

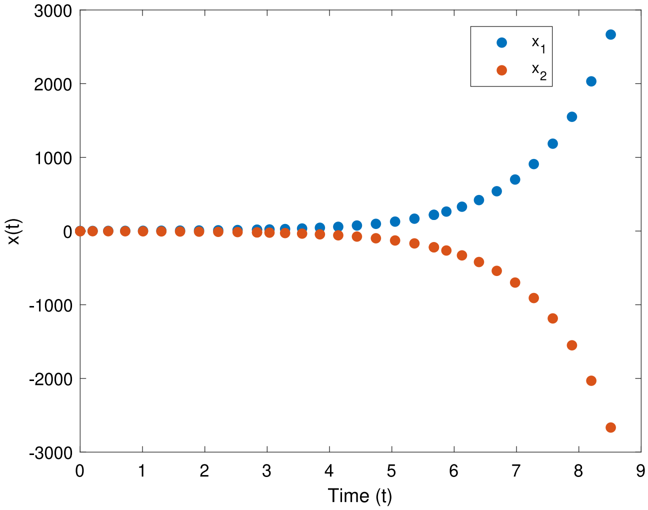

Fig. 2. Trajectories of the unstable discrete subsystem.

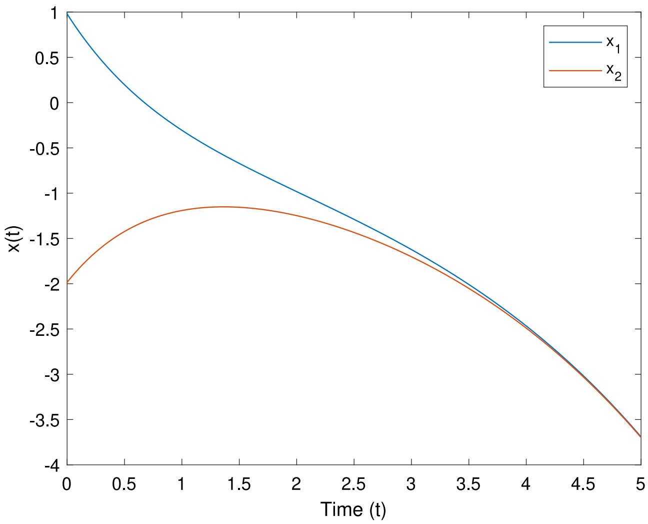

Fig. 3. Trajectories of the unstable continuous subsystem.

For the matrix P satisfying (39) to exist, it is necessary that the matrix $\begin{bmatrix} (A_c - \frac{\gamma}{2}I) & 0 \\ -(I+\mu_{\max}A_d) & -\frac{1}{2\gamma}I \end{bmatrix}$ be Hurwitz. By fixing some feasible γ, we can solve the LMI (37) with respect to P (see [33,34]).

5. Numerical example

Consider the switched system (10) with $A_c = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$ and $A_d = \begin{bmatrix} -1 & -2 \\ -2 & -1 \end{bmatrix}$ which are both unstable, such that $\lambda_c^1 = -3, \lambda_c^2 = 1$ and $\lambda_d^1 = -3, \lambda_d^2 = 1$ (see Figs. 2, 3). Consider $\mu(t_k) = \frac{k}{3k+2}$ for $k = 1, 2, \dots, \infty$, so $\frac{1}{5} \le \mu(t_k) \le \frac{1}{3}$. Figs. 2 and 3 show the unstable trajectories of discrete-time and continuous-time subsystems, respectively. Let $\gamma = 2.5$ (any $\gamma > 2$ is feasible from (39)), and by solving the LMI (37), we get $P = \begin{bmatrix} 5.7177 & 0 \\ 0 & 5.7177 \end{bmatrix}$ for $\alpha = 0.5455$ and

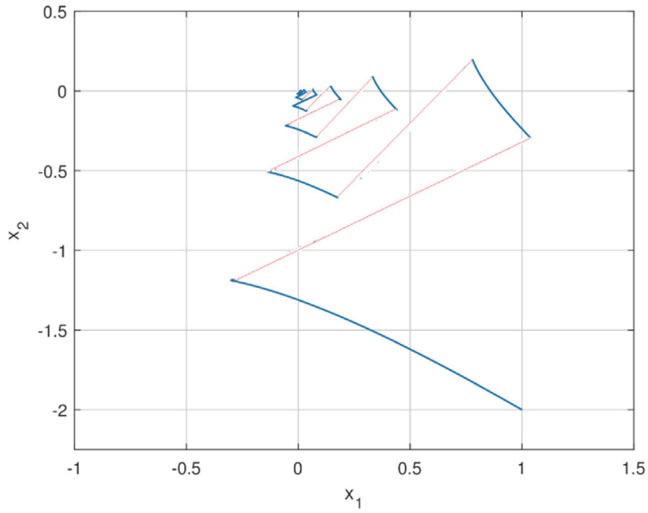

Fig. 4. Stable trajectories by switching. The blue line describes the continuous-time dynamic and the red fine line describes the discrete-time dynamic (the jump). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

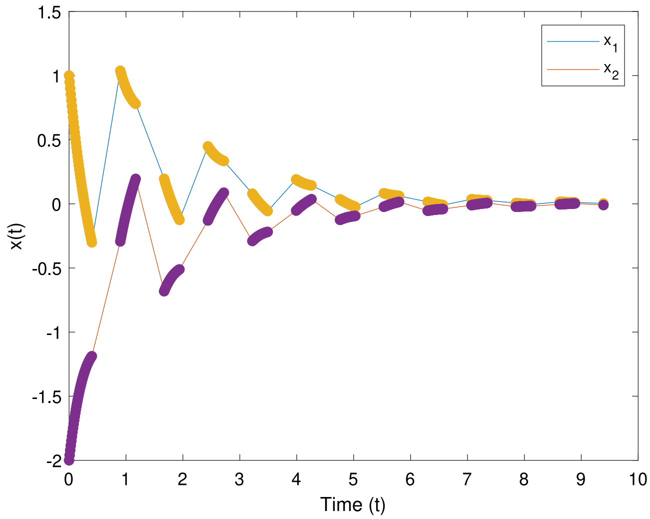

Fig. 5. Stable trajectories by switching. The thick lines represent the continuous-time part and the fine lines represent the discrete-time part (the jump).

$\varepsilon = 4.64$. Let $x_0 = [1 \ -2]^T$. We have $x_0^T(A_c^T P + PA_c)x_0 = -148.6602$, and $x_0^T(A_d^T P + PA_d + \mu_{\max} A_d^T PA_d)x_0 = 50.3872$. So we will activate at first the continuous-time subsystem. According to the switching rule (22), the switched system is stabilized as shown in Figs. 4 and 5.

6. Conclusion

In this paper, a necessary and sufficient condition for the quadratic stabilizability of a particular class of switched systems evolving on a non-uniform time domain is derived by introducing time scales theory. The system switches between an unstable continuous-time subsystem and an unstable discrete-time subsystem with variable discrete step sizes. It has been shown that by choosing properly the switching rule according to specific state conditions, the switched system can be quadratically stabilized. A numerical example to illustrate the switching rule was provided.

CRediT authorship contribution statement

Fatima Z. Taousser: Writing – original draft, Deriving the result, Doing numerical computations, Doing simulations. **Seddik M. Djouadi:** Deriving the result, Doing numerical computations, Writing – review & editing. **Kevin Tomsovic:** Deriving the result, Writing – Review & Editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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